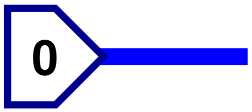
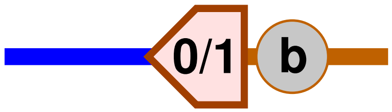
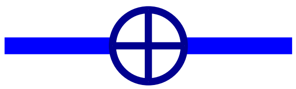
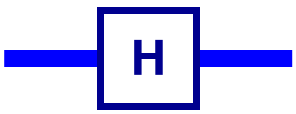
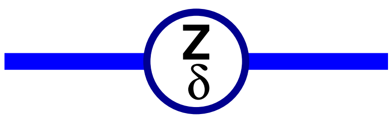
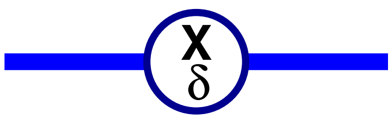
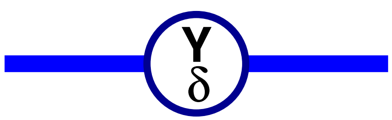
IQI 04, Seminar 5

- Continuous one-qubit rotations.
- Application: Refocusing.
- Conditional rotations.
- Phase kick-back.
- The rotation-angle problem.

E. “Manny” Knill: knill@boulder.nist.gov

Summary of One-Qubit Gates

Gate picture	Symbol	Matrix form
	$\mathbf{prep}(\mathsf{0})$	
	$\mathbf{meas}(Z{\mapsto}b)$	
	$\mathbf{not}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
	$\mathbf{had}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
	$\mathbf{Z}_\delta$	$\begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$
	$\mathbf{X}_\delta$	$\begin{pmatrix} \cos(\delta/2) & -i\sin(\delta/2) \\ -i\sin(\delta/2) & \cos(\delta/2) \end{pmatrix}$
	$\mathbf{Y}_\delta$	$\begin{pmatrix} \cos(\delta/2) & -\sin(\delta/2) \\ \sin(\delta/2) & \cos(\delta/2) \end{pmatrix}$

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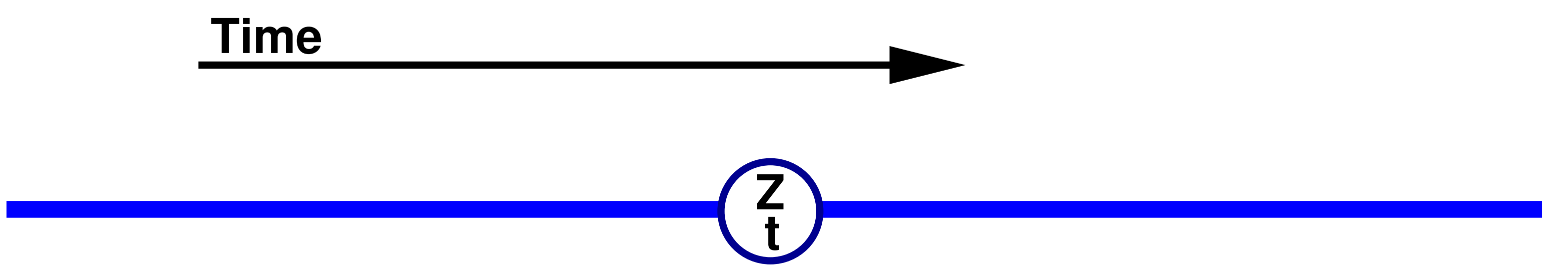

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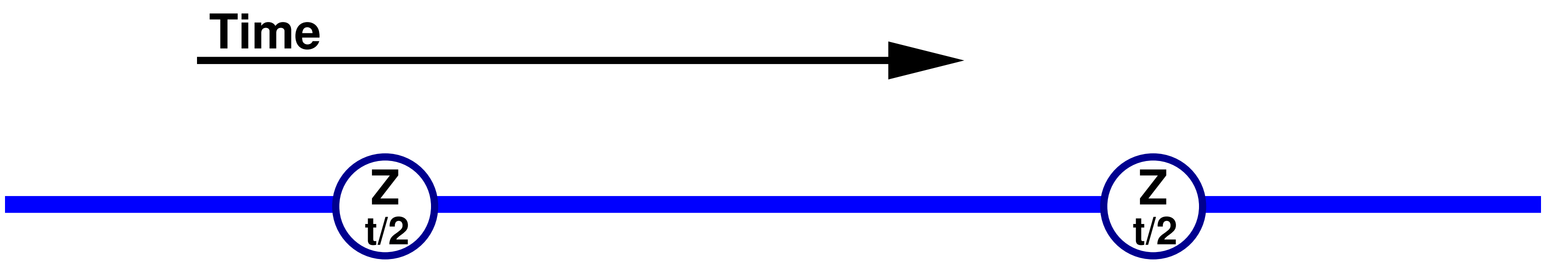

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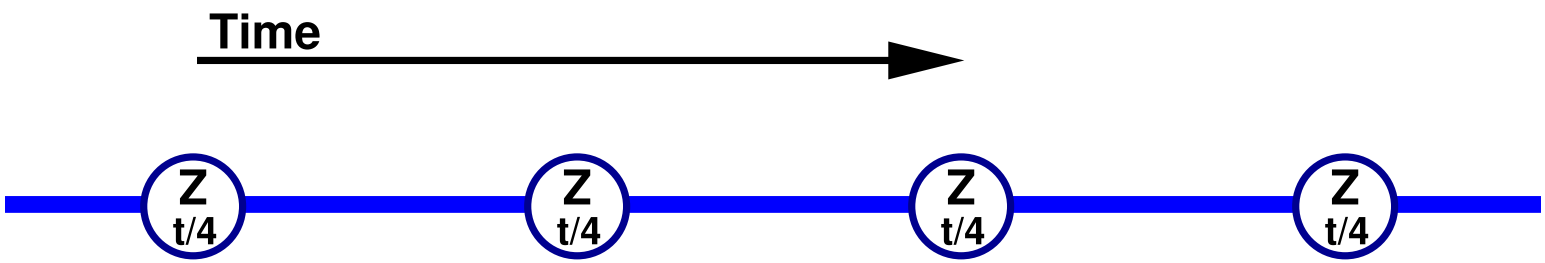

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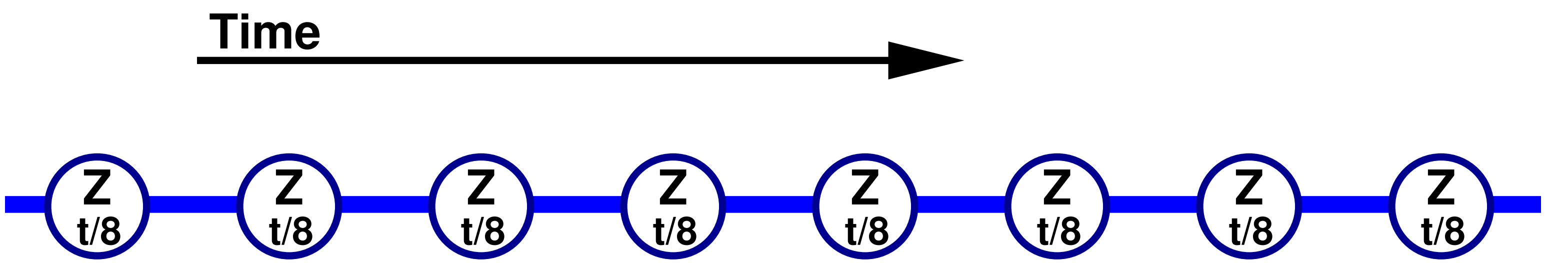

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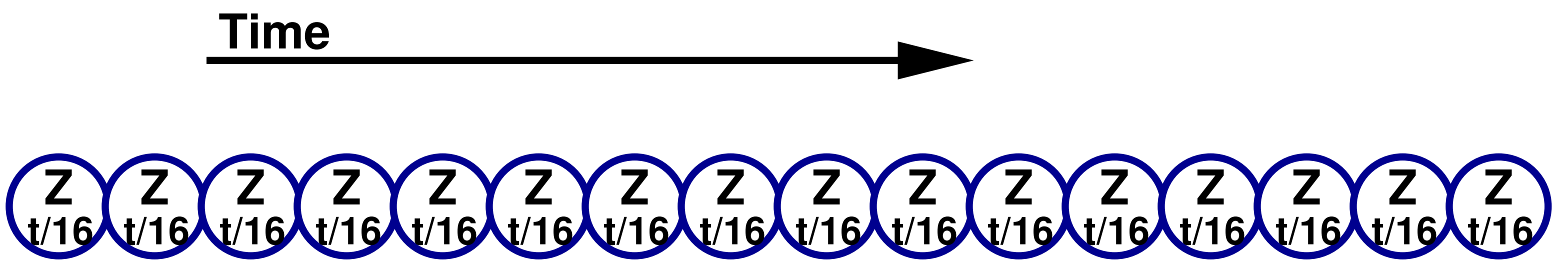

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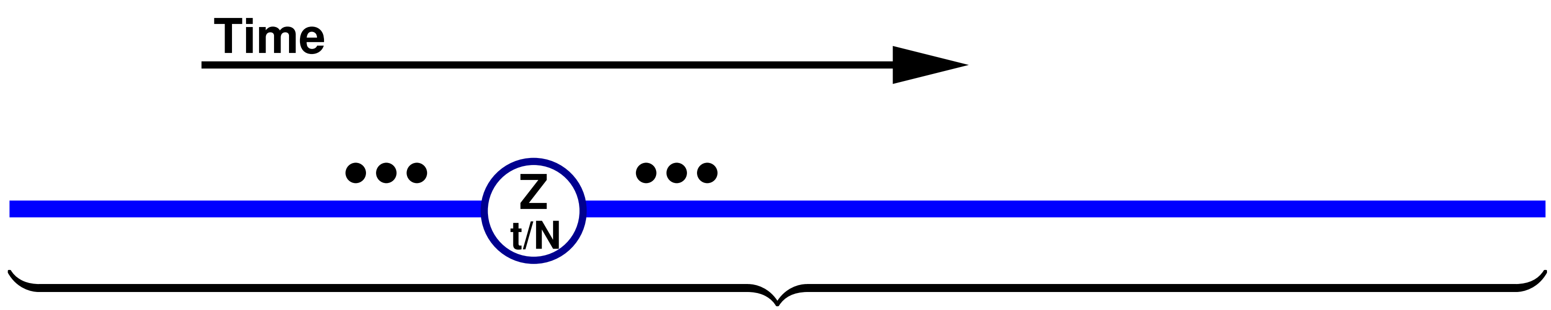

N times, take $N \to \infty$

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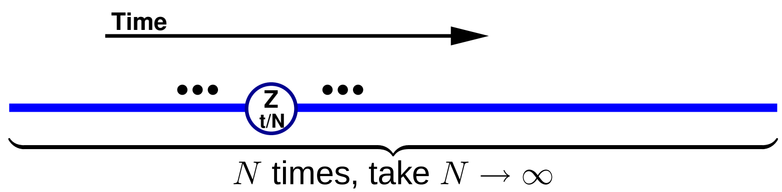

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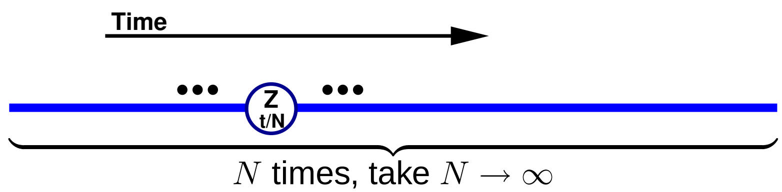

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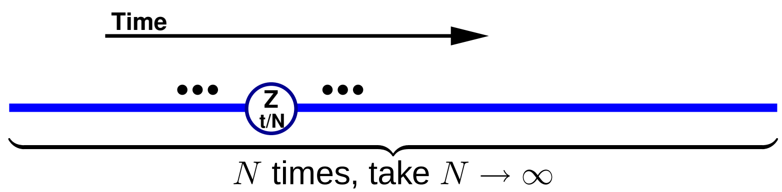

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$$\mathbf{Z}_t = \lim_{N\to\infty}\left(\mathbb{1} - i\frac{t}{N}\begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\right)^N = e^{-i(\sigma_z/2)t}$$

where $\sigma_z/2 \doteq \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}$ is the *generator* for $\mathbf{Z}$ rotations.

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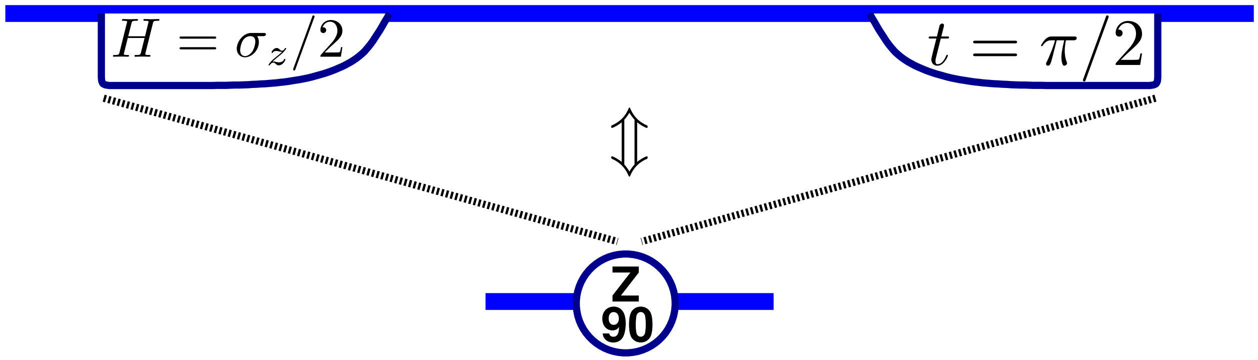

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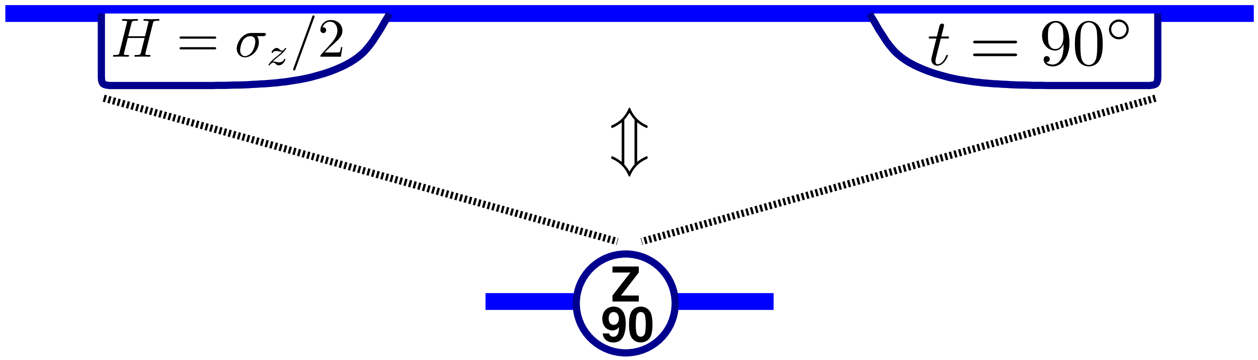

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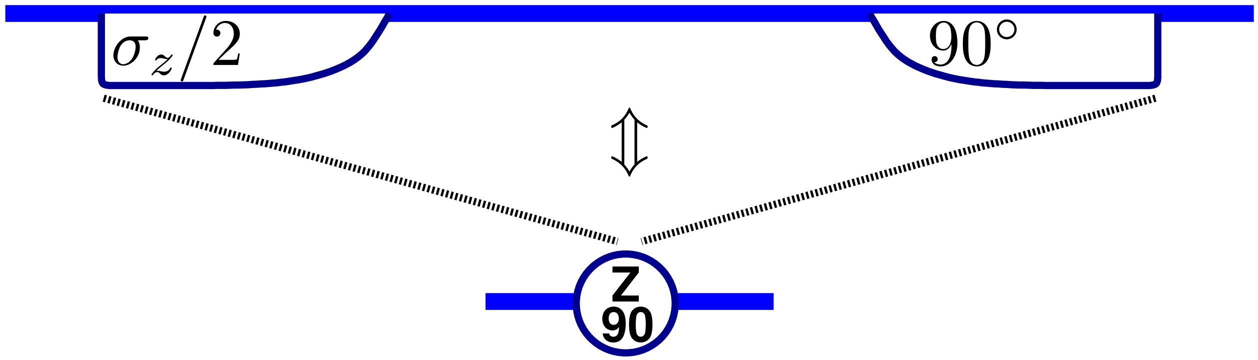

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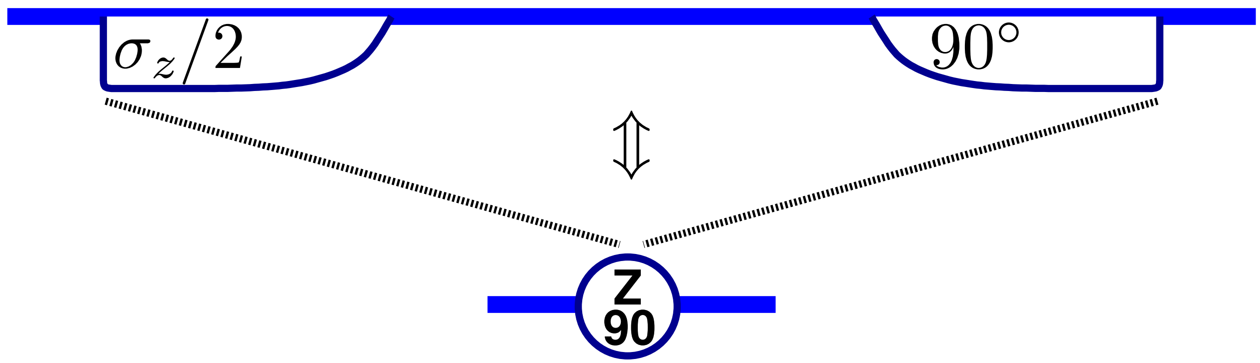

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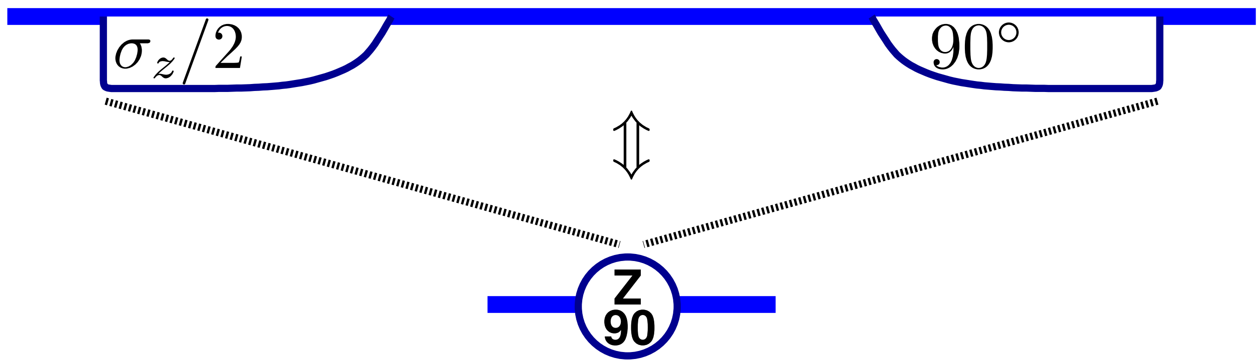

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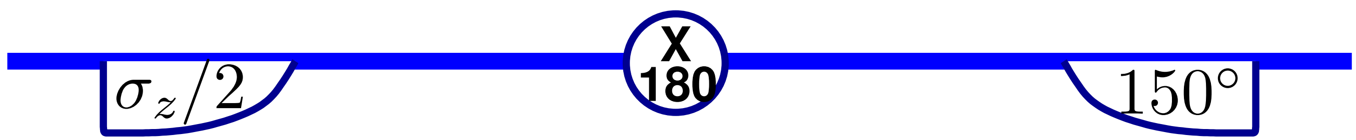

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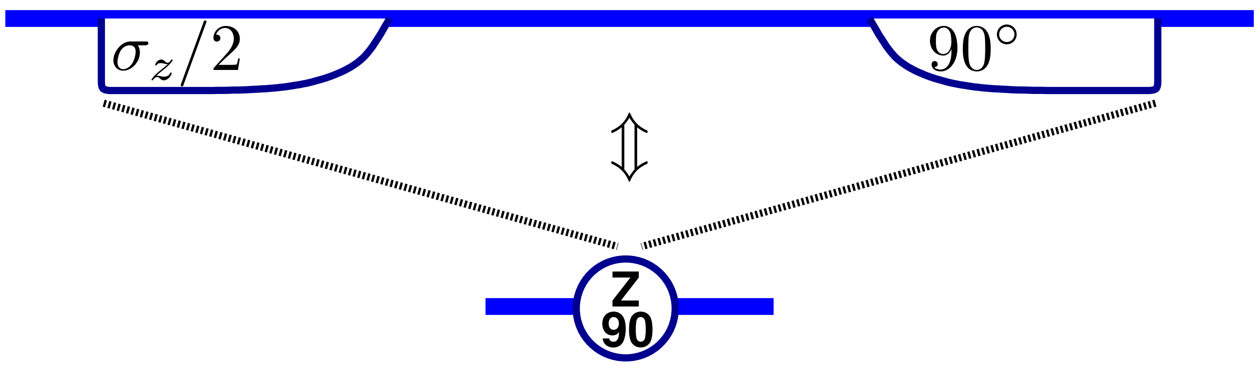

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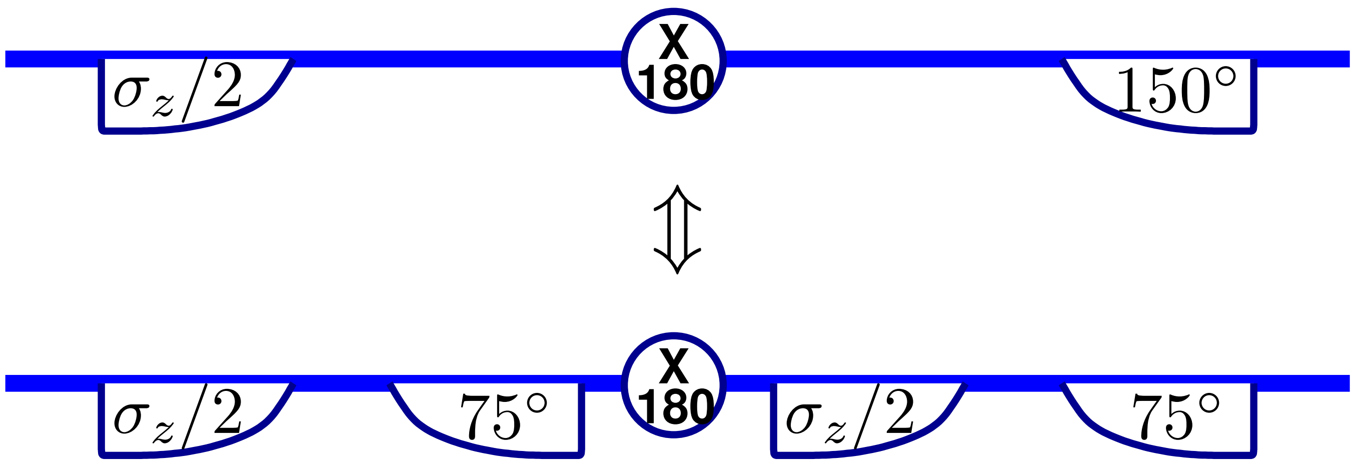

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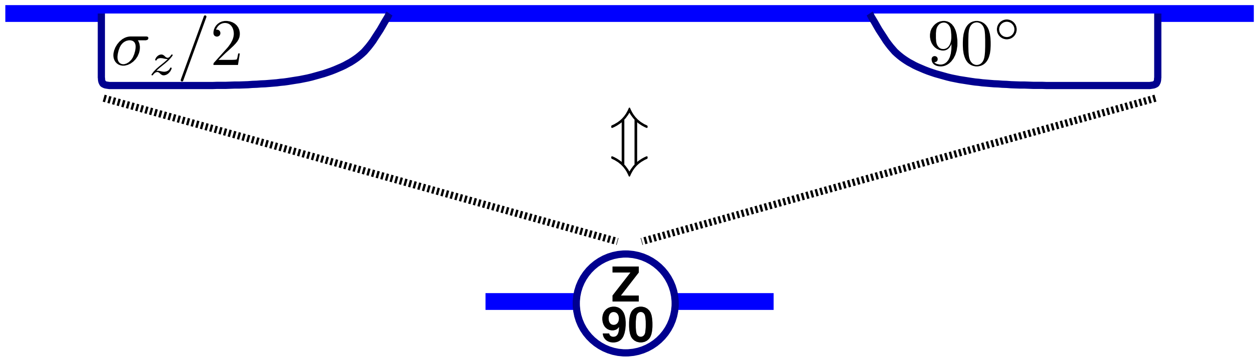

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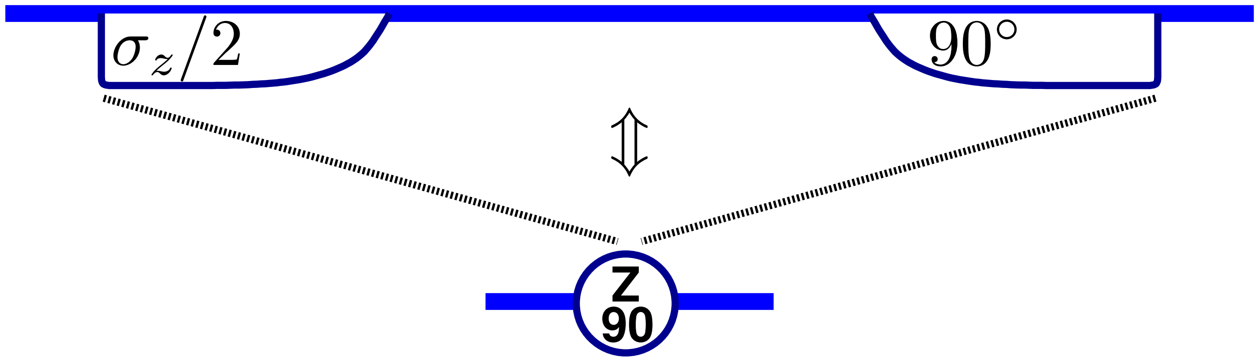

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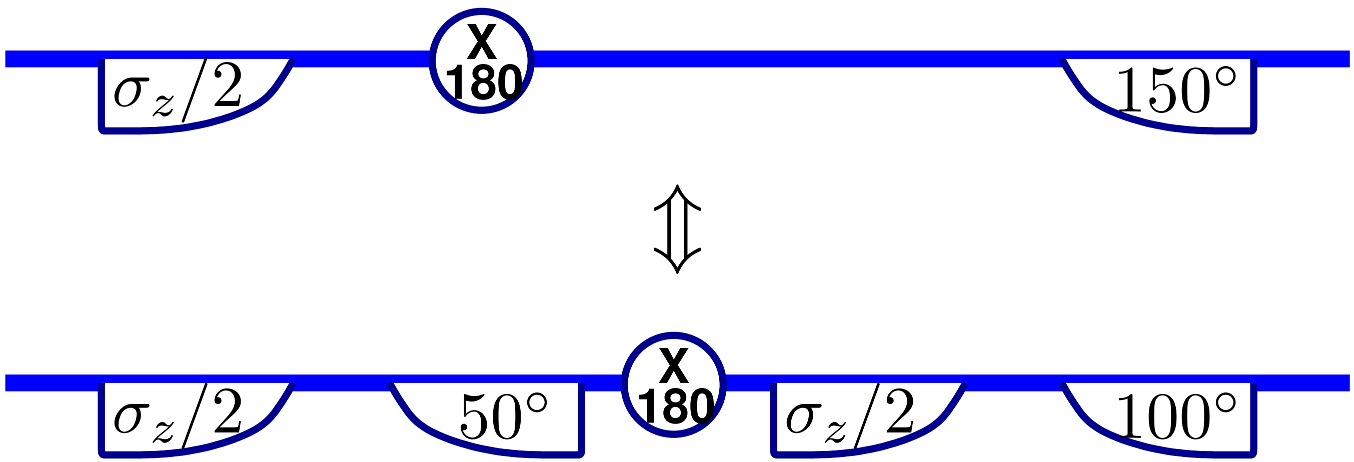

Continuous Rotations Around Any Axis

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  - Use $e^X = \mathbb{1} + X + X^2/2! + X^3/3! + X^4/4! + \ldots$
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    4. $UX^kU^\dagger = U\,XX\ldots\,U^\dagger = UXU^\dagger\,UXU^\dagger\,\ldots = (UXU^\dagger)^k$.

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- Why is $e^{-i(\hat{u}\cdot\vec{\sigma}/2)\delta} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\hat{u}\cdot\vec{\sigma}$?
  - Or use:
    1. $e^{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}} = \begin{pmatrix} e^a & 0 \\ 0 & e^b \end{pmatrix}$
    2. So $e^{-i(\sigma_z/2)\delta} = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_z$.
    3. Choose $\hat{v}$ and ϵ so that $\mathbf{rot}(\hat{v},\epsilon)\sigma_z\mathbf{rot}(\hat{v},-\epsilon) = \hat{u}\cdot\vec{\sigma}$.
    4. $UX^kU^\dagger = U\,XX\ldots U^\dagger = UXU^\dagger\,UXU^\dagger\,\ldots = (UXU^\dagger)^k$.
    5. $Ue^XU^\dagger = U(\mathbb{1}+X+X^2/2+\ldots)U^\dagger$
       $= \mathbb{1} + UXU^\dagger + (UXU^\dagger)^2/2 + \ldots = e^{UXU^\dagger}$.

Continuous Rotations Around Any Axis

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…for time t.

- The Hamiltonian is applied, or is part of the qubit's dynamics.
  - Note units: Energy units are angular frequency, $\hbar = 1$.

Example: Spin $1/2$ Qubit

- Spin $1/2$ in oriented space: One particle in a superposition of the states "up" ($|\uparrow\rangle$) and "down" ($|\downarrow\rangle$).

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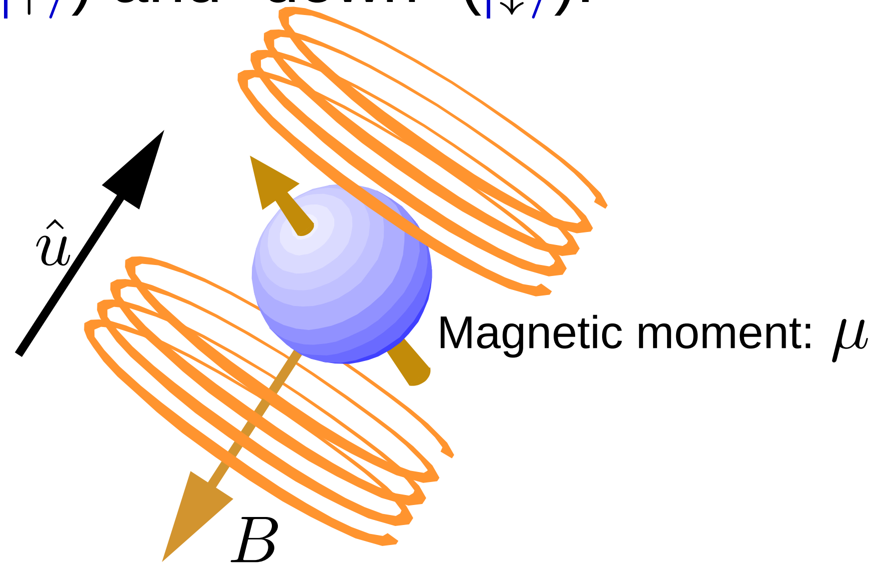

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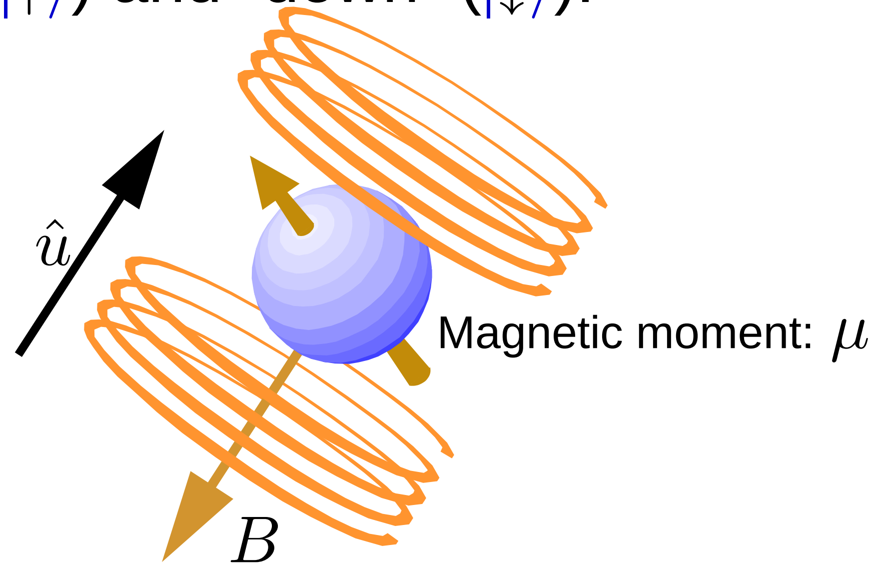

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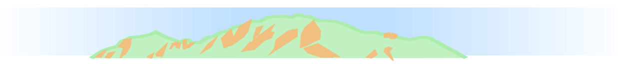
- How to remove the effect of ϵJ_z dynamics with ϵ unknown?



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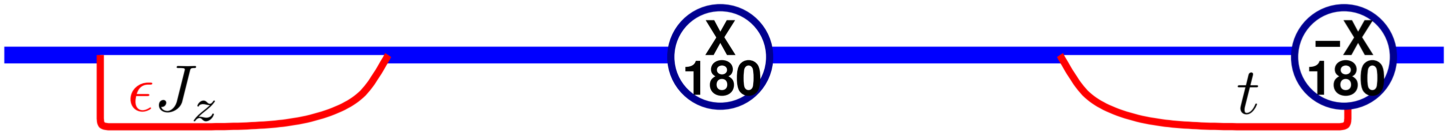

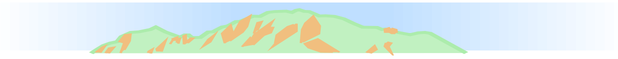

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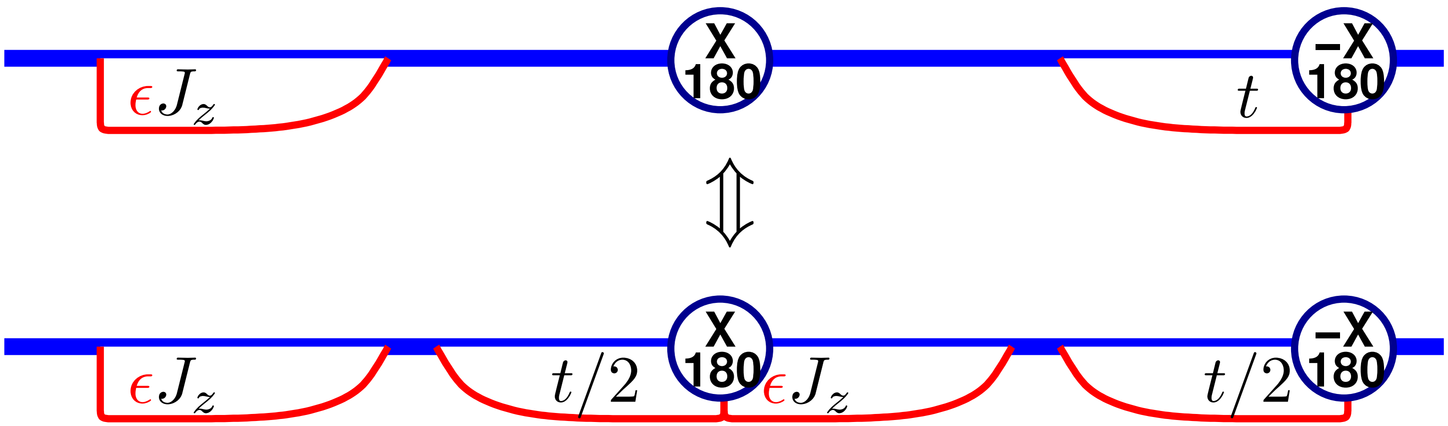

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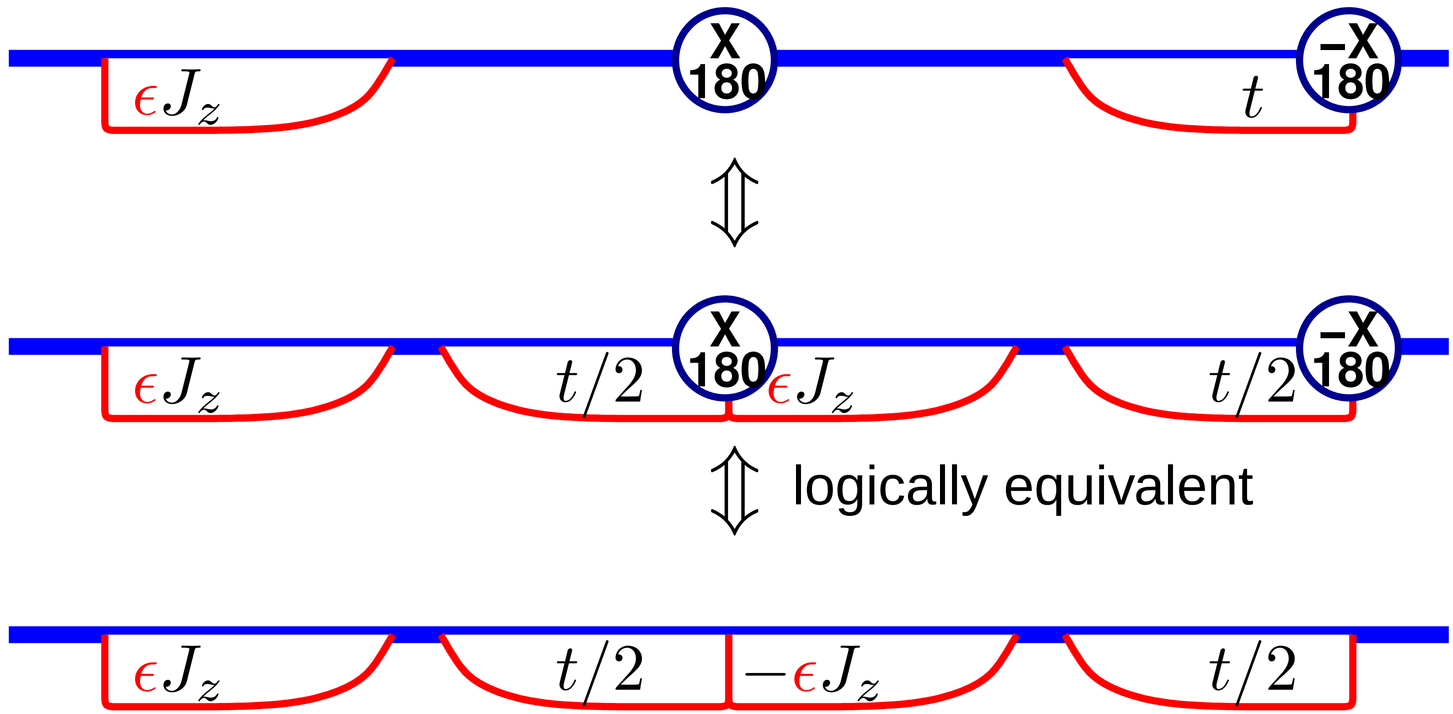

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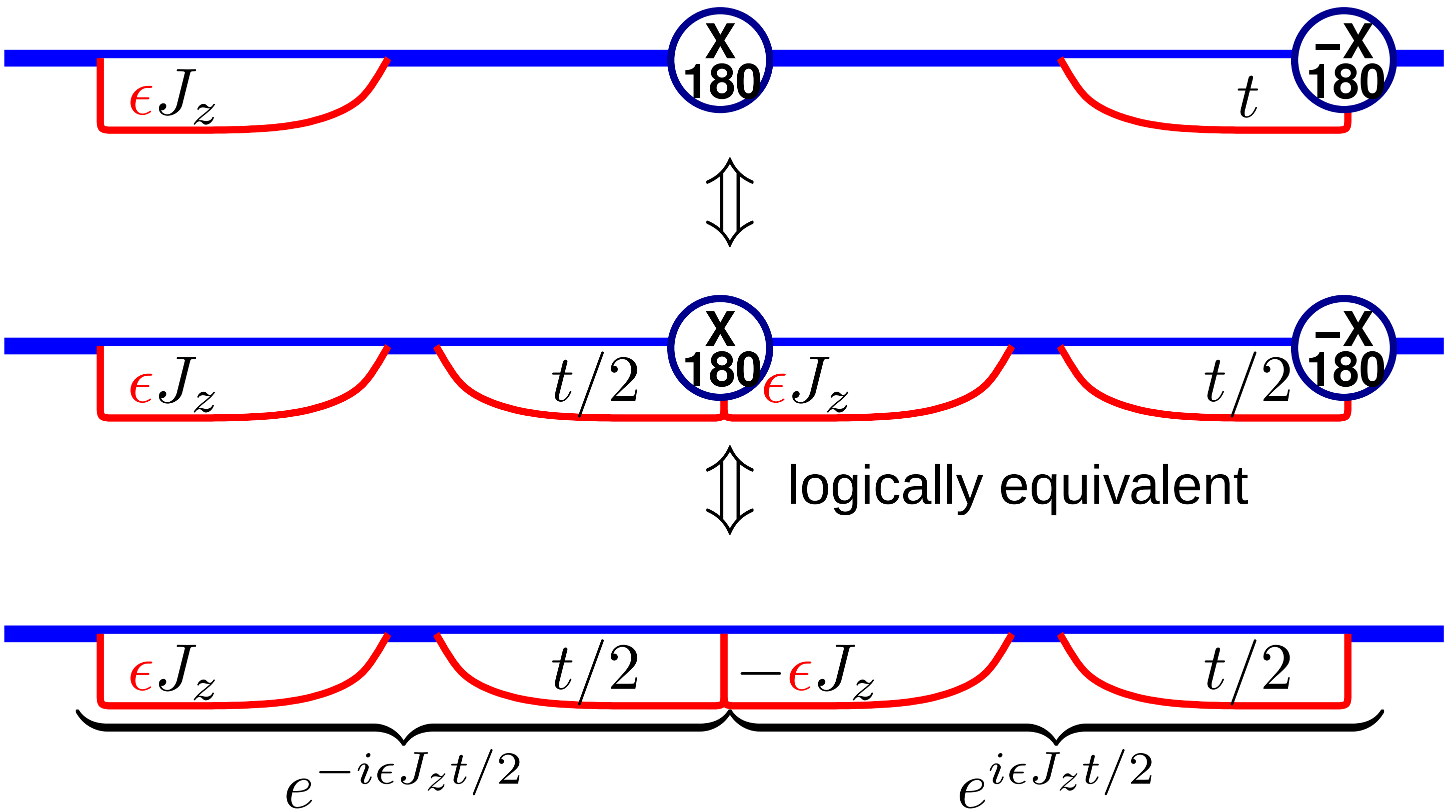

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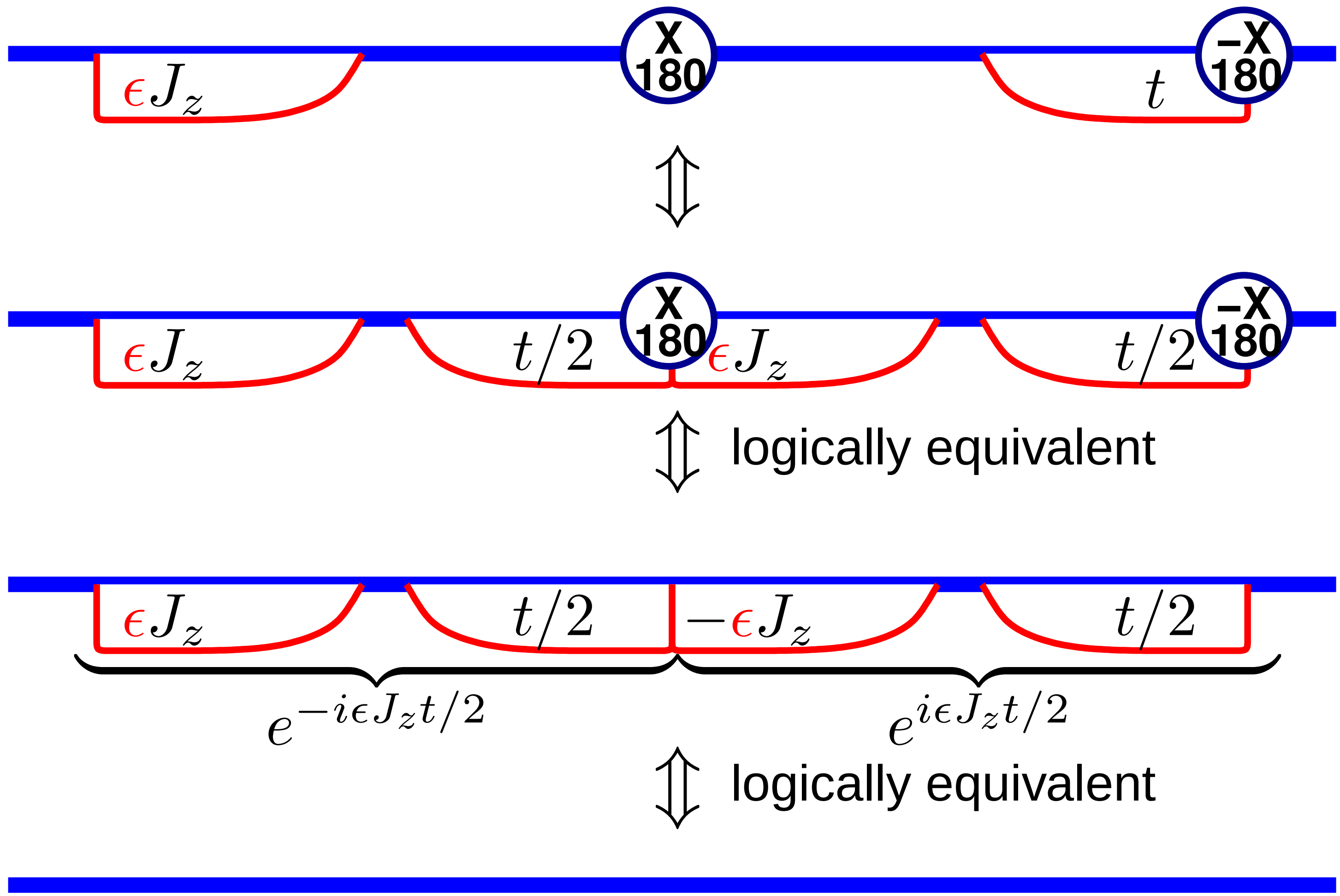

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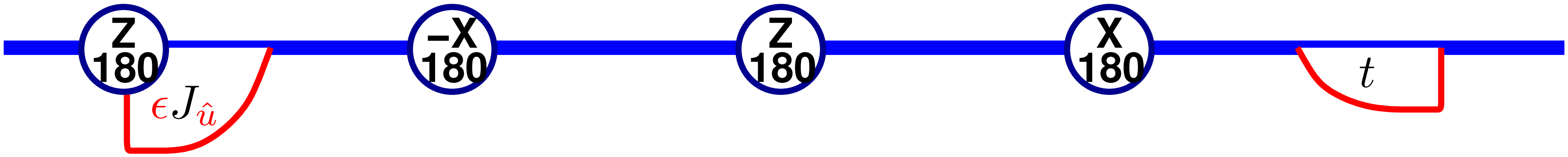

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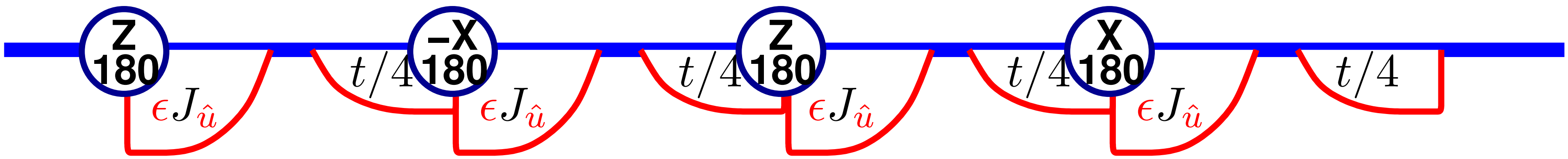

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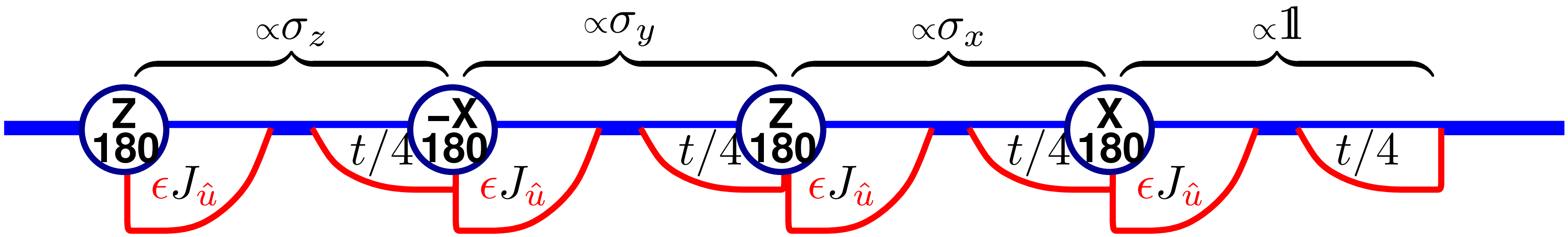

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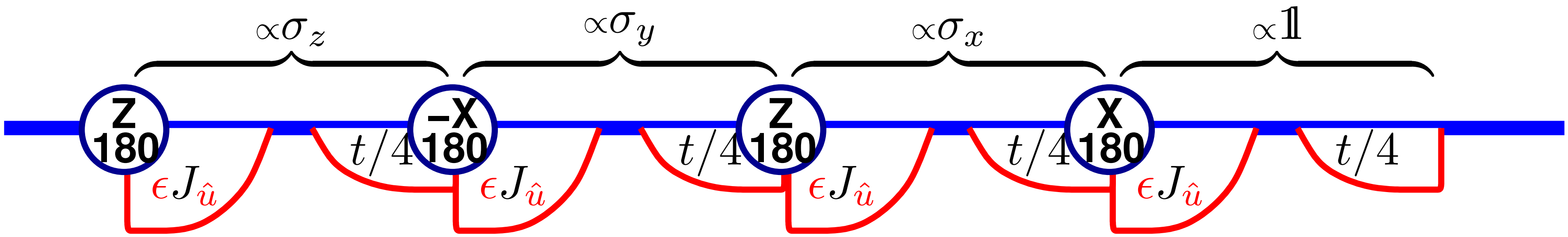
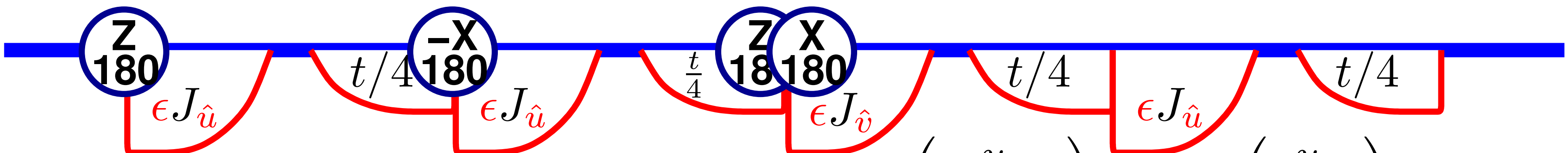


$\Updownarrow$ logically equivalent



$$\hat{v} = \begin{pmatrix} u_x \\ -u_y \\ -u_z \end{pmatrix} \quad \hat{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

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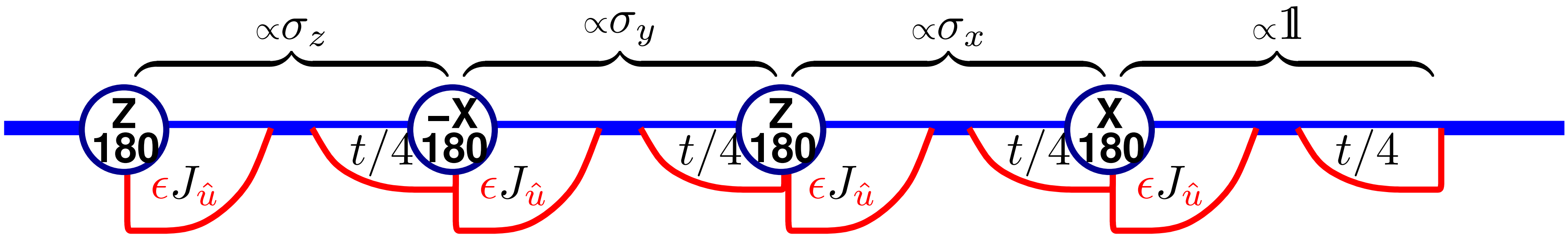

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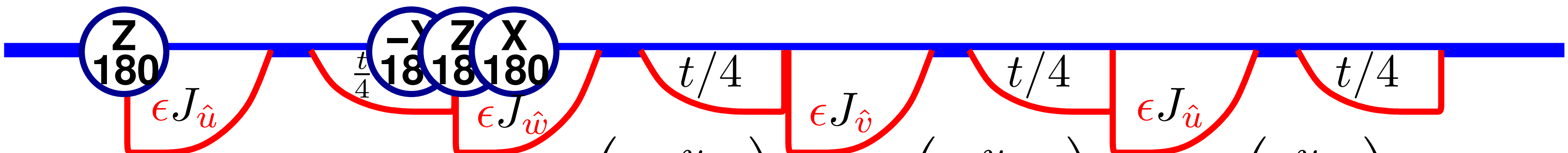

$$\hat{w} = \begin{pmatrix} -u_x \\ u_y \\ -u_z \end{pmatrix} \quad \hat{v} = \begin{pmatrix} u_x \\ -u_y \\ -u_z \end{pmatrix} \quad \hat{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

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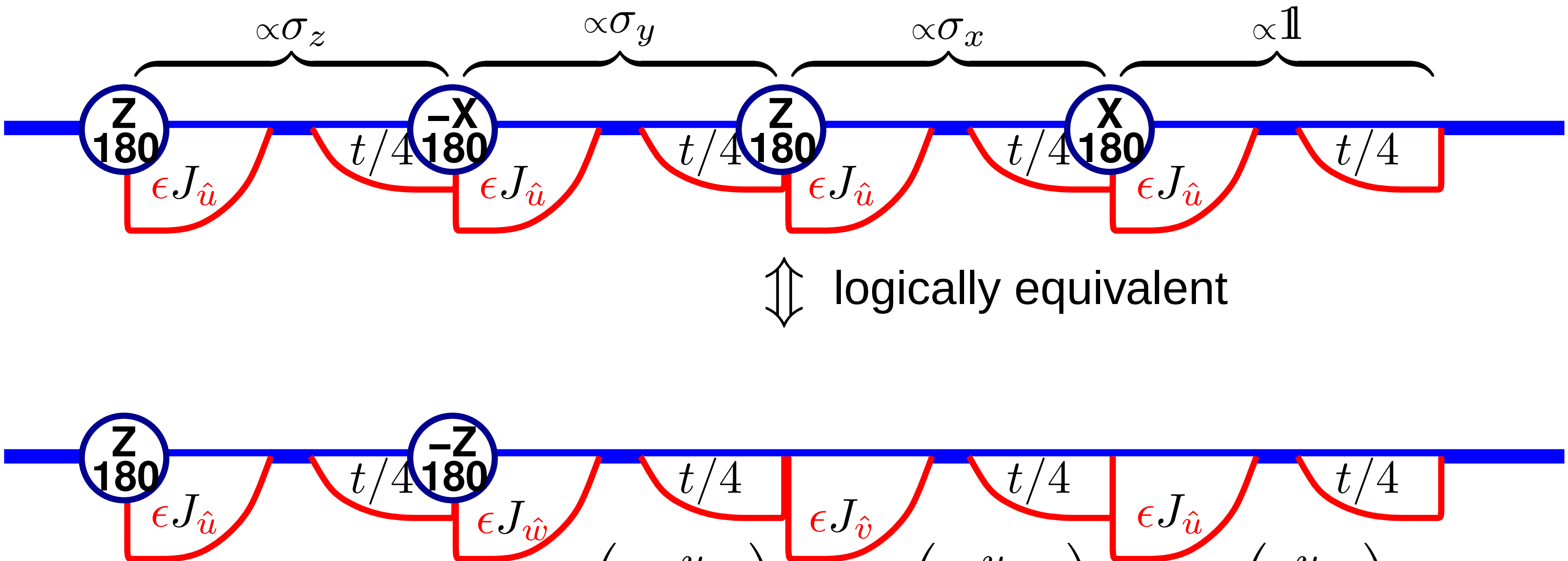

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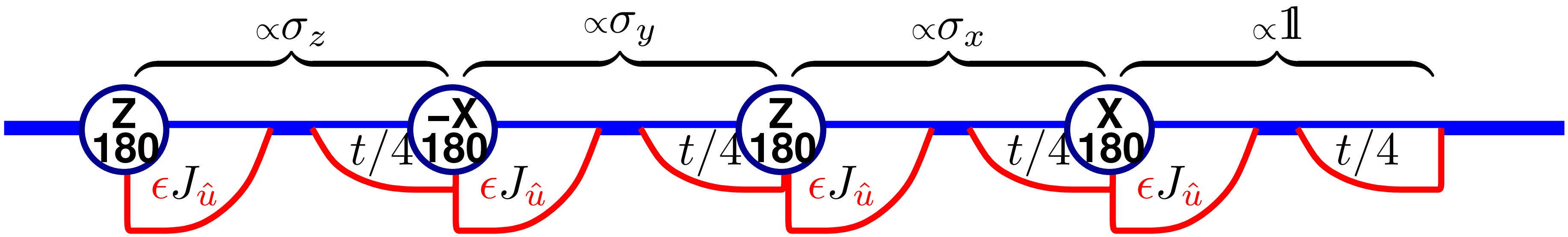

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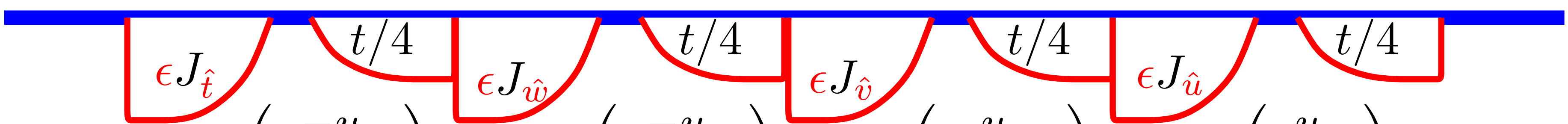

$$\hat{t} = \begin{pmatrix} -u_x \\ -u_y \\ u_z \end{pmatrix} \quad \hat{w} = \begin{pmatrix} -u_x \\ u_y \\ -u_z \end{pmatrix} \quad \hat{v} = \begin{pmatrix} u_x \\ -u_y \\ -u_z \end{pmatrix} \quad \hat{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$$

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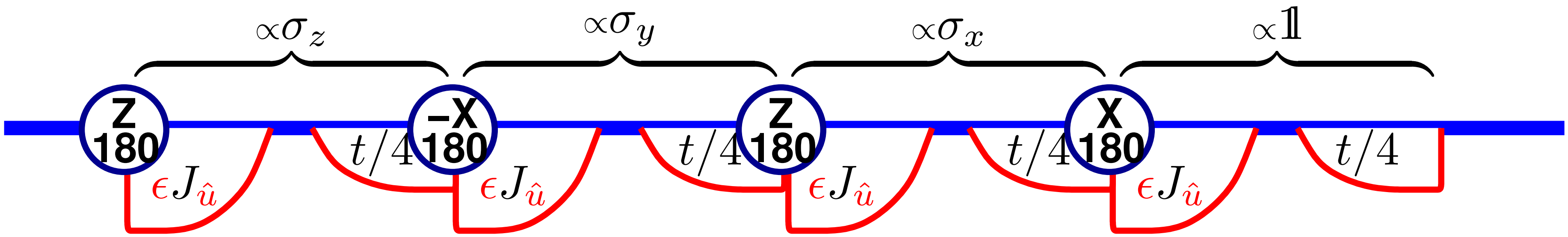

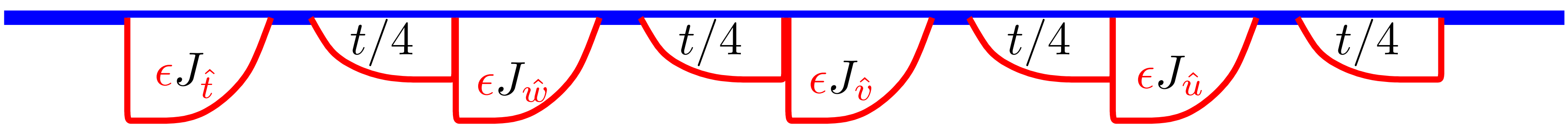

$$e^{-i(\epsilon t/4)(-u_x J_x - u_y J_y + u_z J_z)}$$

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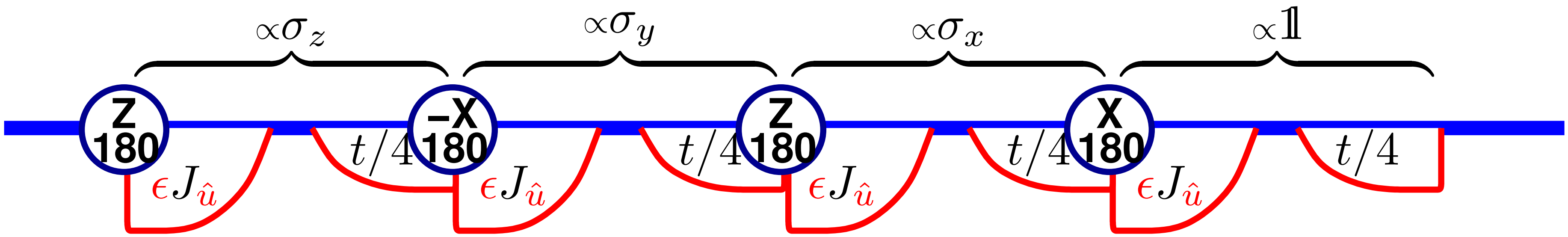

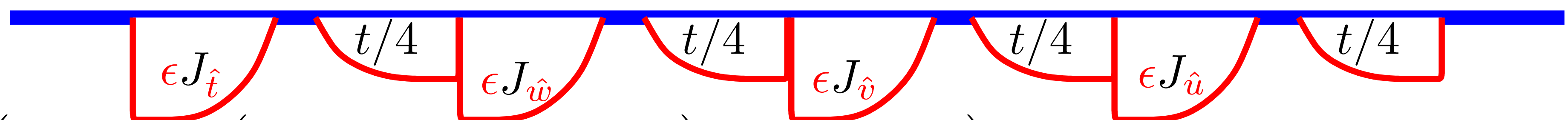

$$\Big(\mathbb{1} - i(\epsilon t/4)\Big(-u_x J_x - u_y J_y + u_z J_z\Big) + O((\epsilon t/4)^2)\Big)$$
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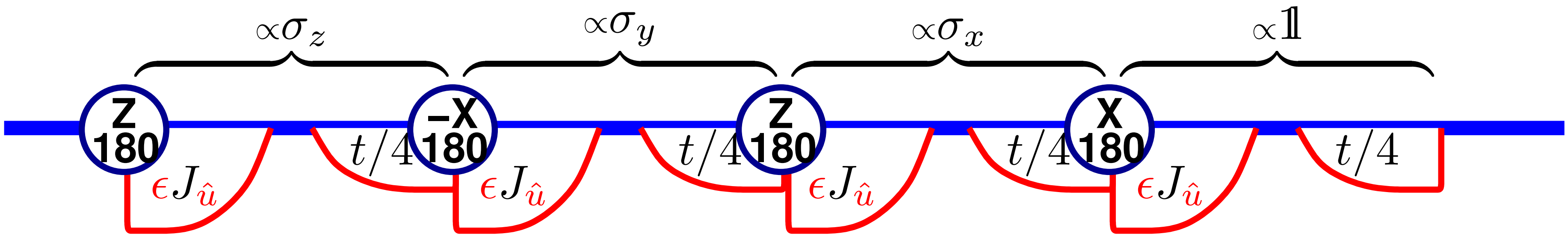

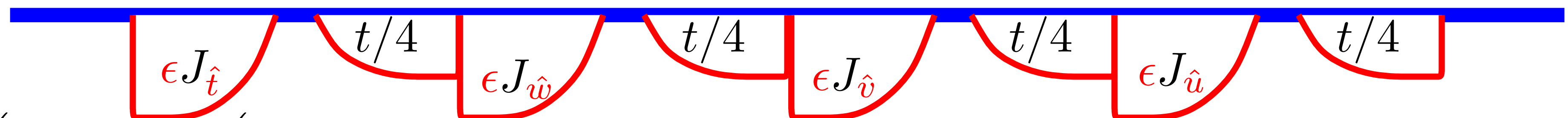

$$\Big(\mathbb{1} - i(\epsilon t/4)\Big(-u_xJ_x - u_yJ_y + u_zJ_z$$
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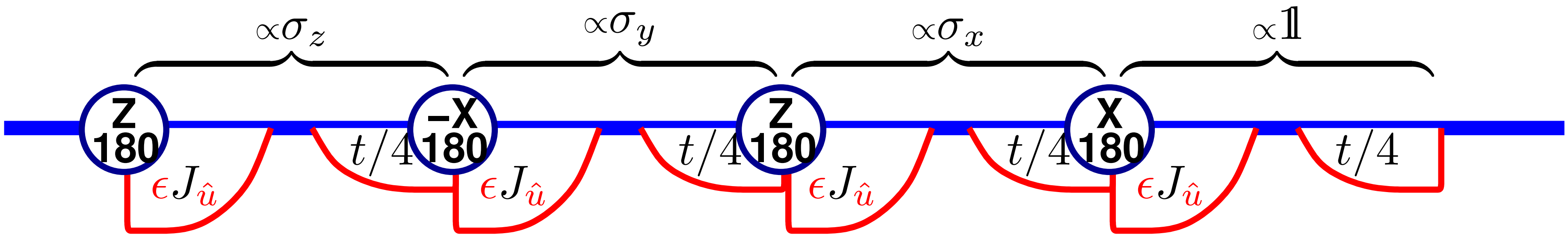
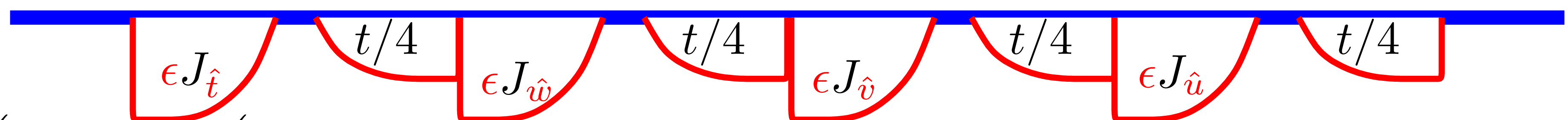


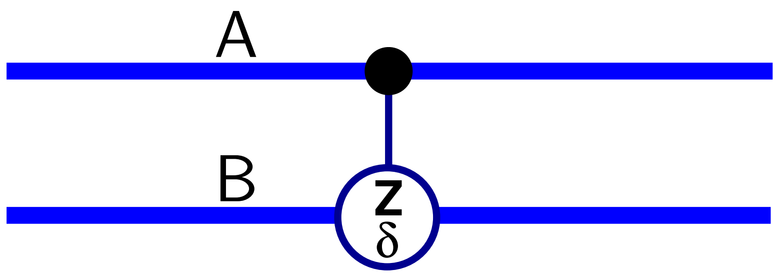


$$\begin{aligned}\Big(\mathbb{1} - i(\epsilon t/4)\Big(&-u_xJ_x - u_yJ_y + u_zJ_z\\ &-u_xJ_x + u_yJ_y - u_zJ_z\\ &u_xJ_x - u_yJ_y - u_zJ_z\\ &u_xJ_x + u_yJ_y + u_zJ_z\Big) + O(4(\epsilon t/4)^2)\Big)\end{aligned}$$

$\Updownarrow$ logically approximately equivalent up to $O(4(\epsilon t/4)^2)$

Conditional Z-Rotations

- Implementation of the conditional $\mathbf{Z}_\delta$ gate, $\mathbf{cZ}_\delta{}^{(\mathsf{AB})}$.



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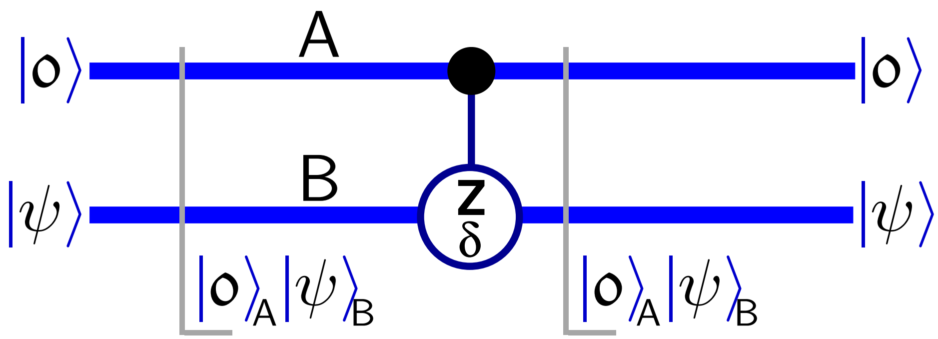

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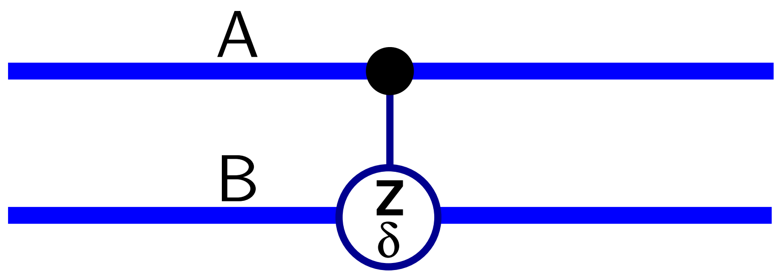

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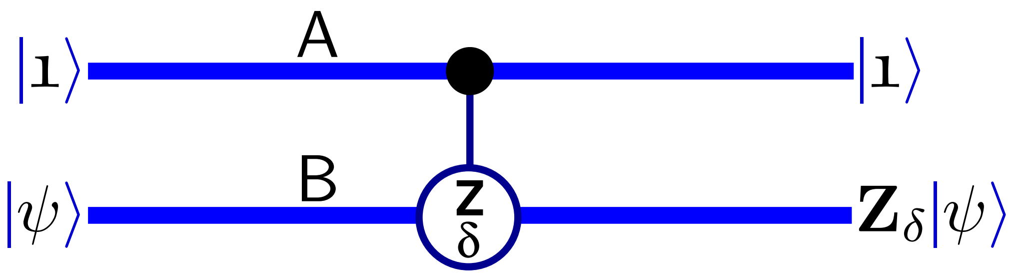

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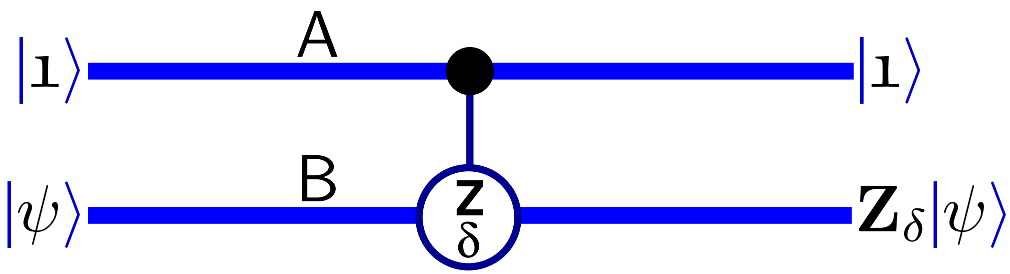

- Implementation with two cnots. $\Updownarrow$

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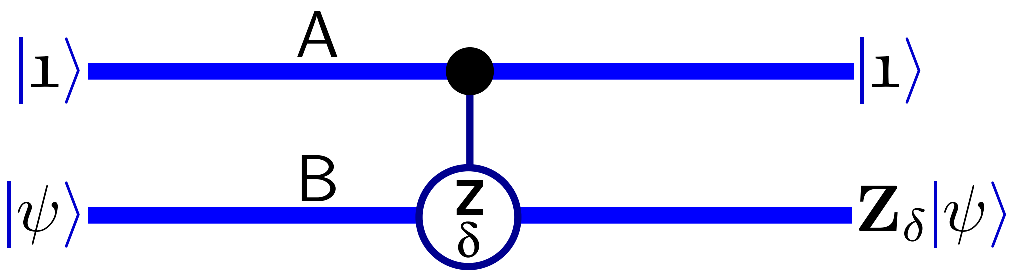

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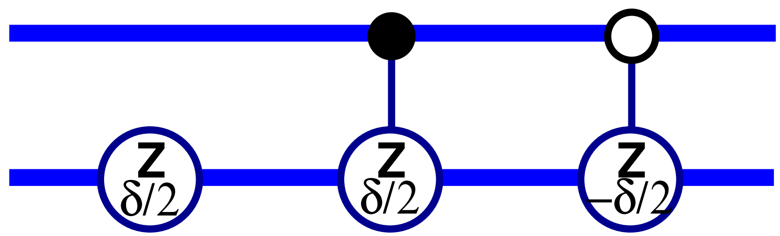

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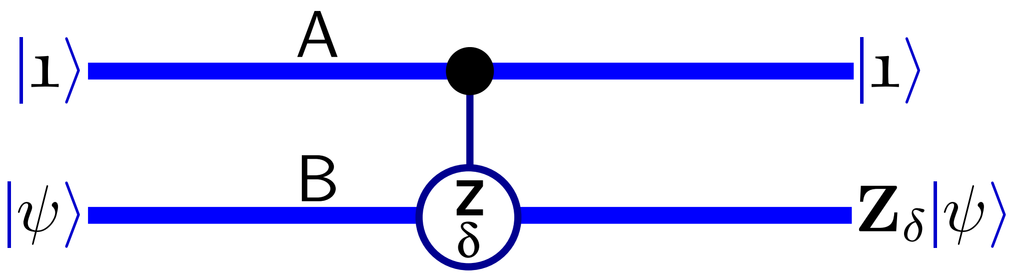

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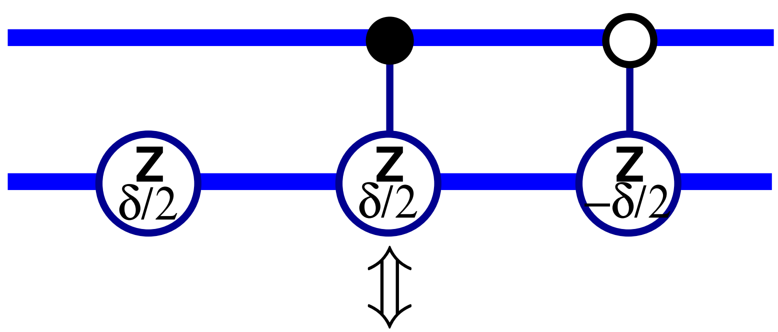

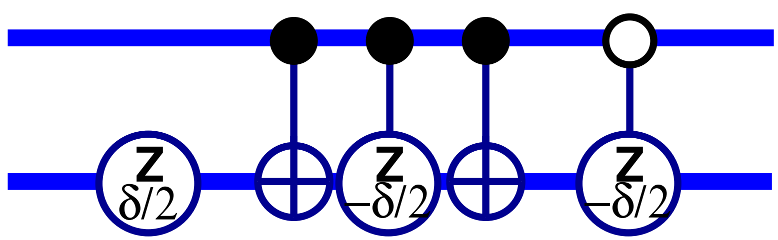

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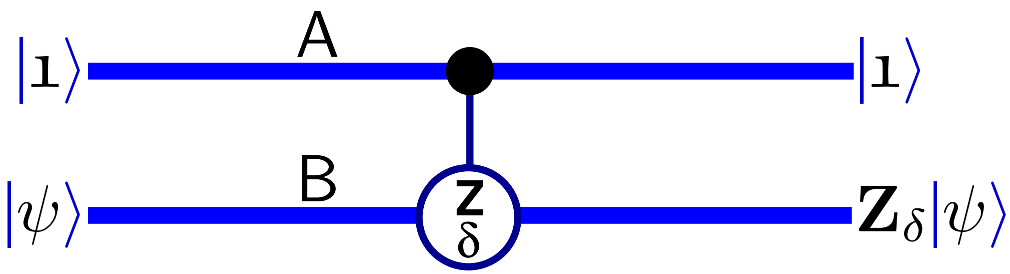

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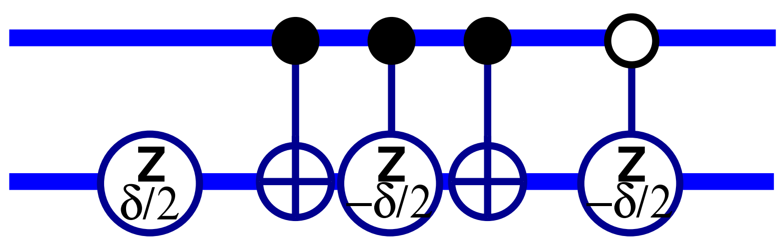

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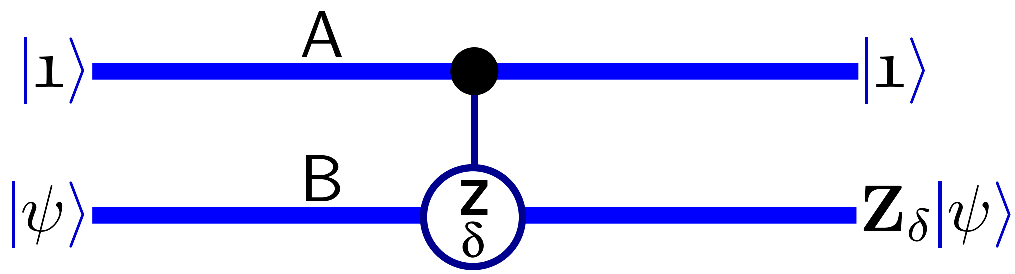

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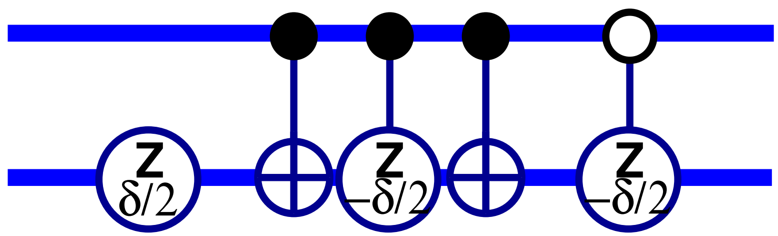

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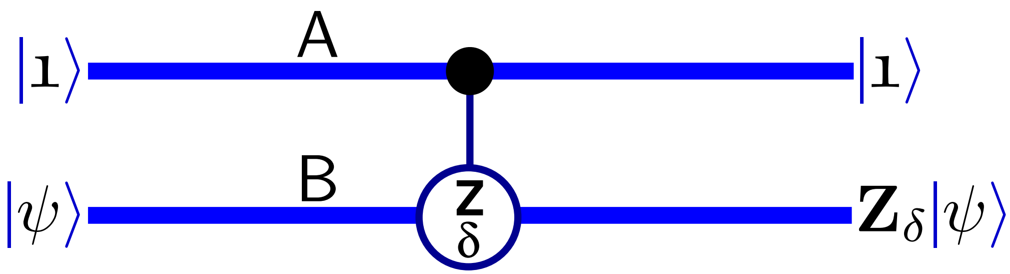

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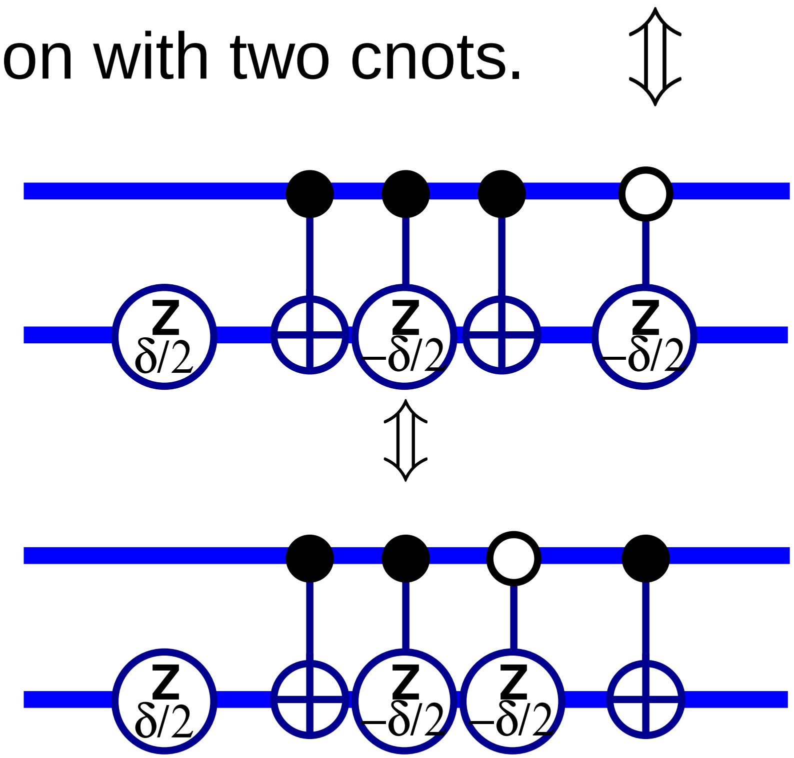

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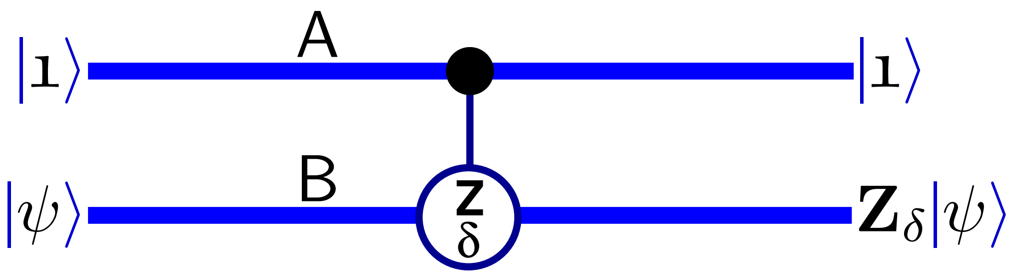

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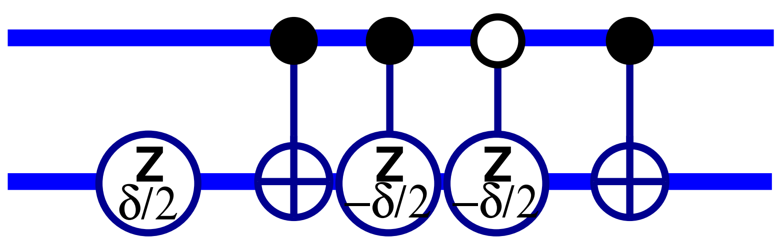

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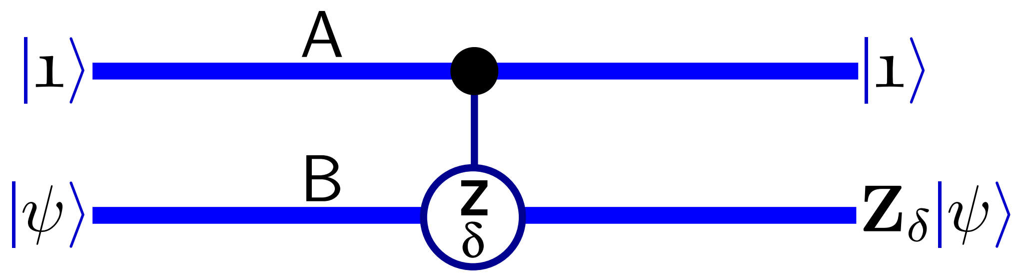

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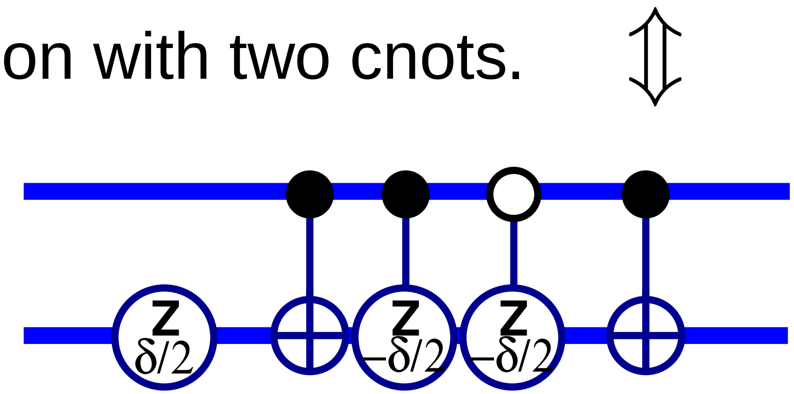

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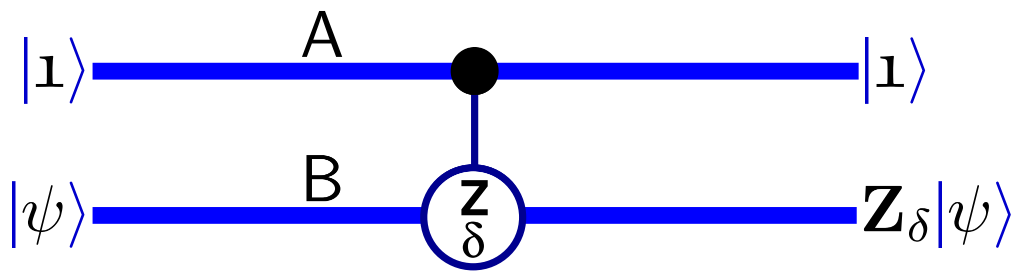

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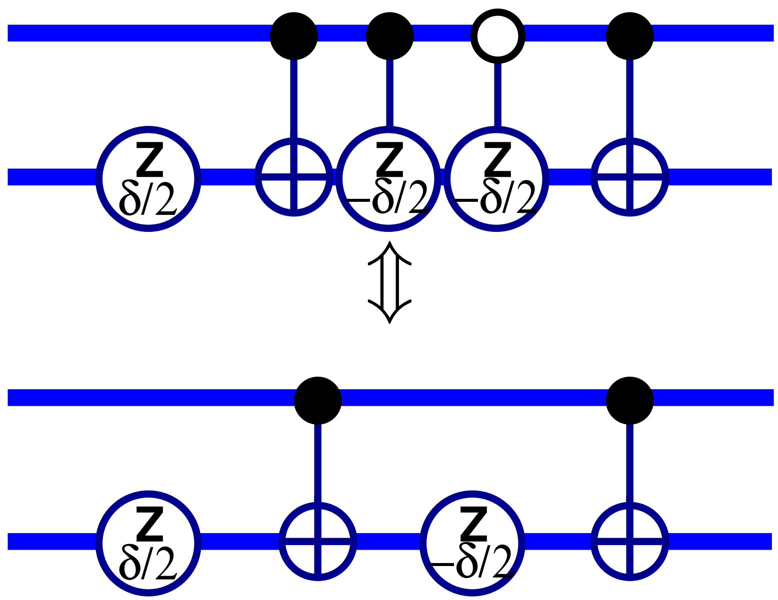

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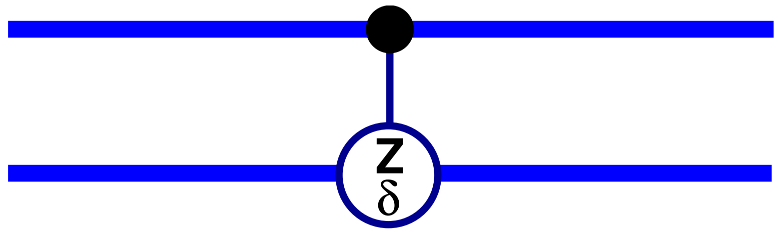
- Conditional rotations conditionally kick back phases.



$$\mathbf{Z}_\delta = \begin{pmatrix} e^{-i\delta/2} & 0 \\ 0 & e^{i\delta/2} \end{pmatrix}$$

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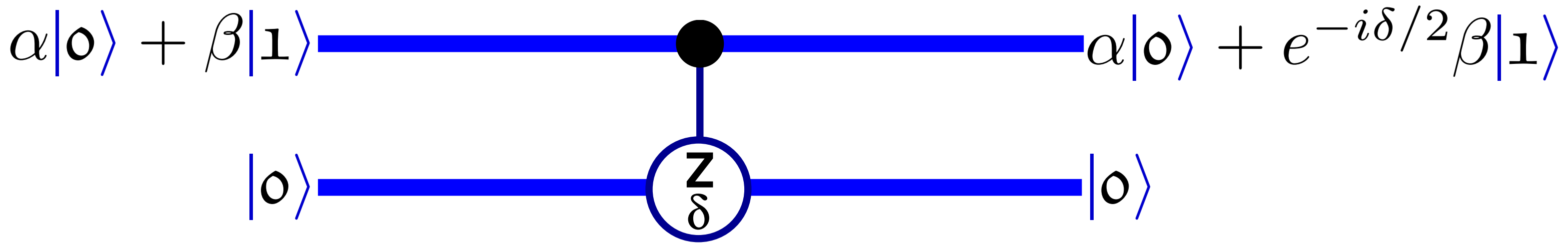

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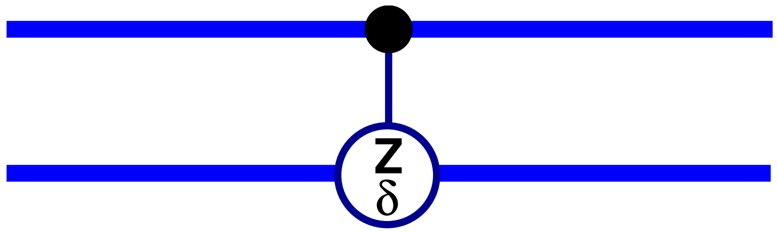

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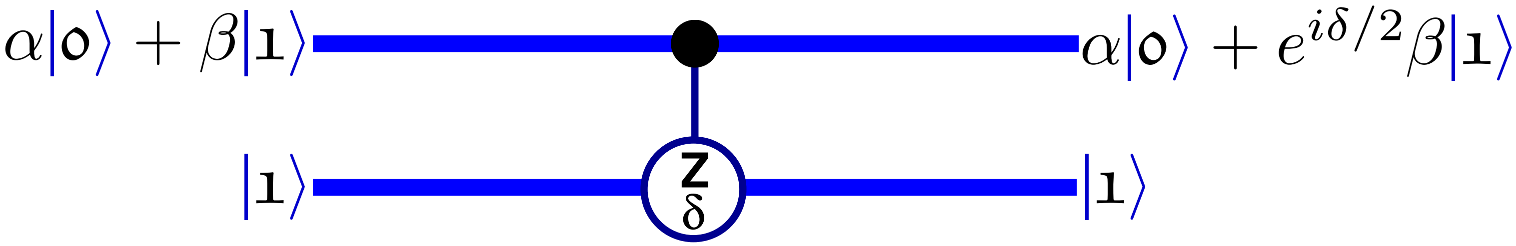

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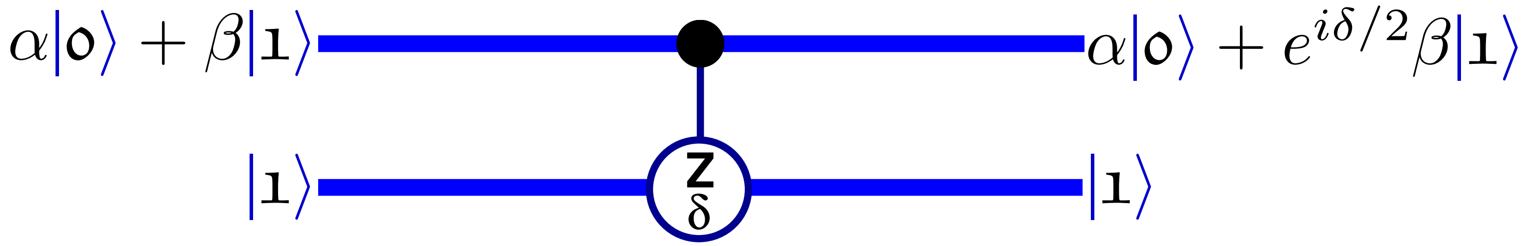

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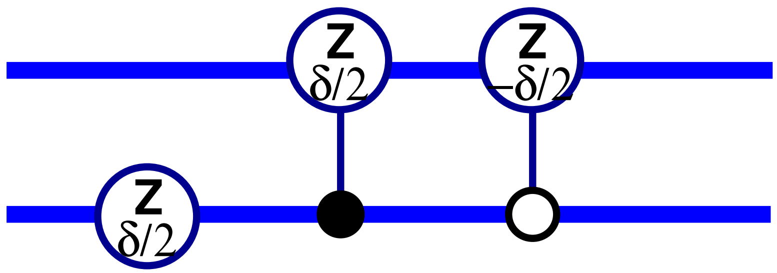

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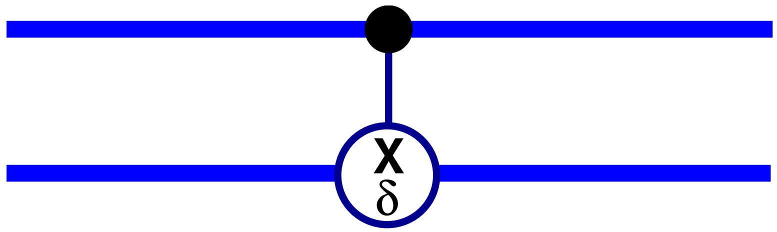

$$\mathbf{X}_\delta = \cos(\delta/2)\mathbb{1} - i\sin(\delta/2)\sigma_x$$

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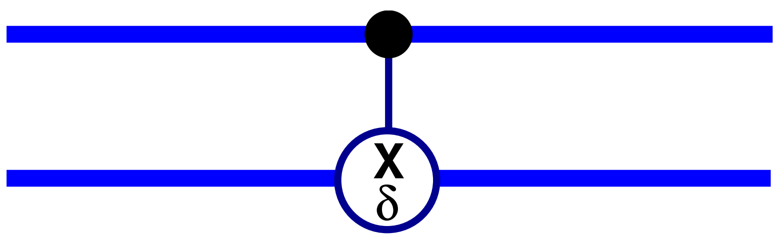
$\alpha|0\rangle + \beta|1\rangle$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$



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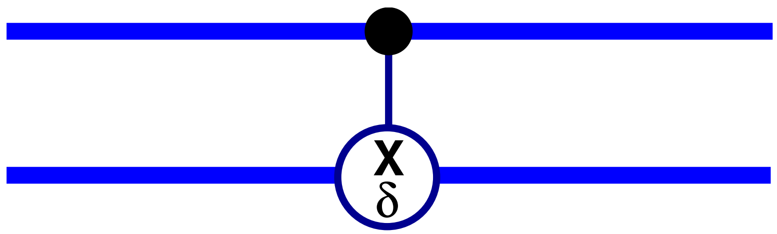

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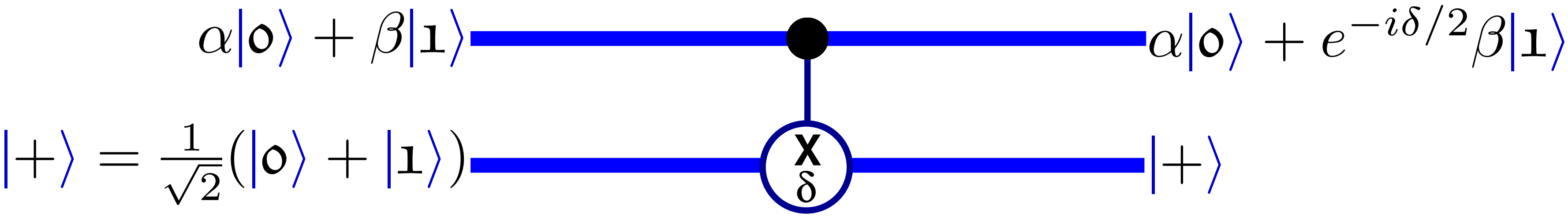

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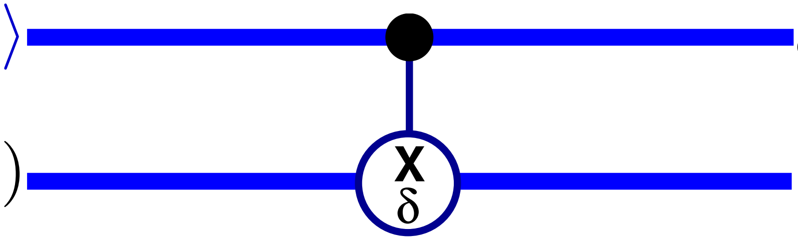

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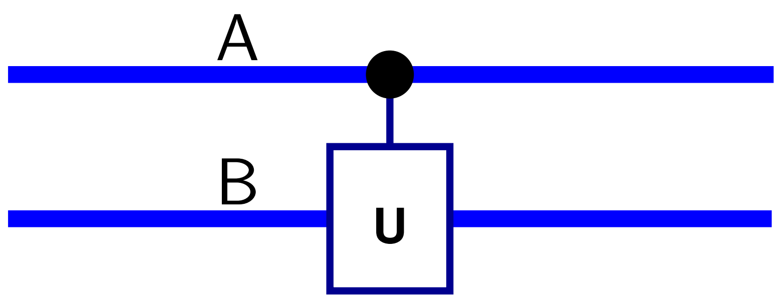

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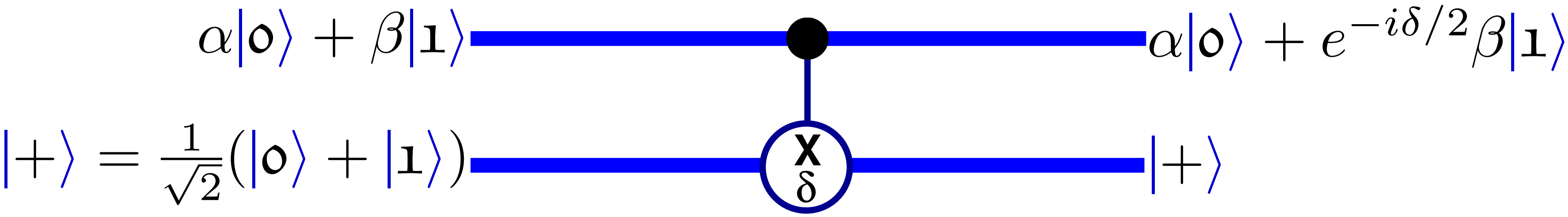

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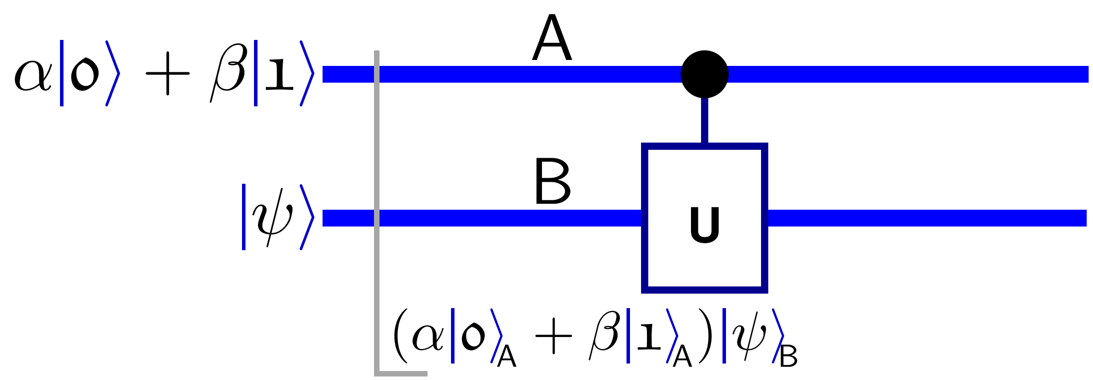

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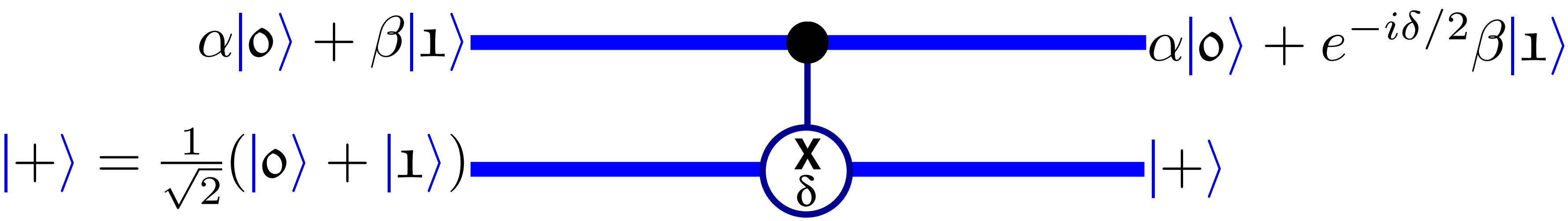

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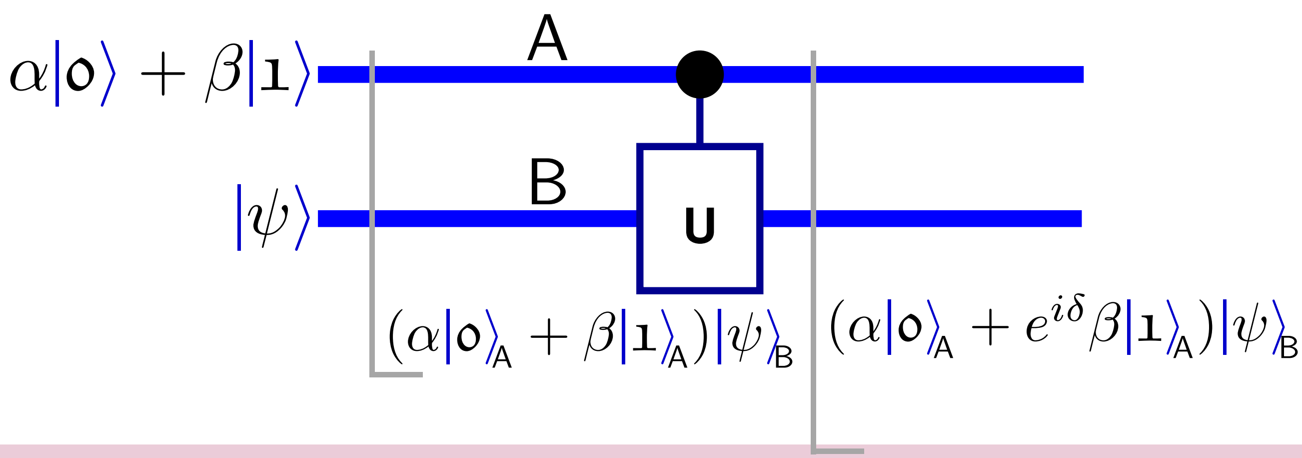

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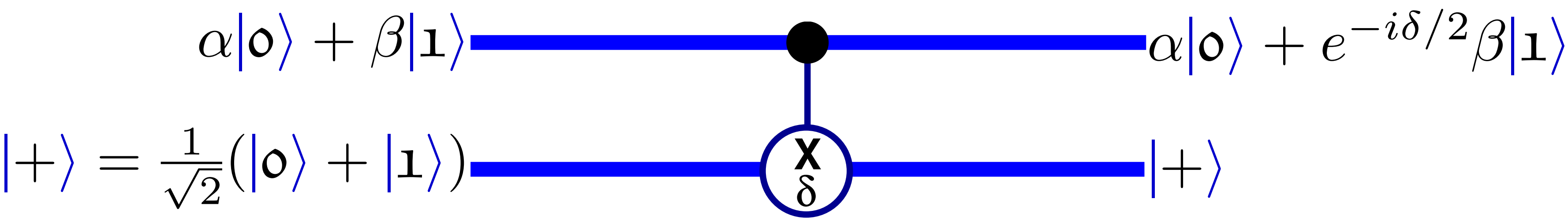

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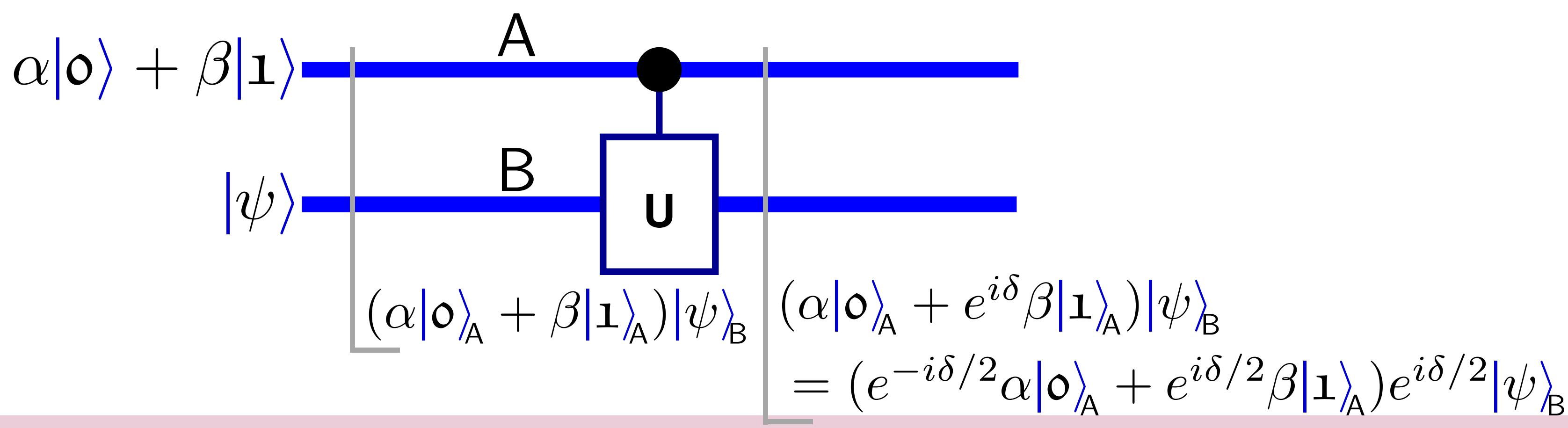

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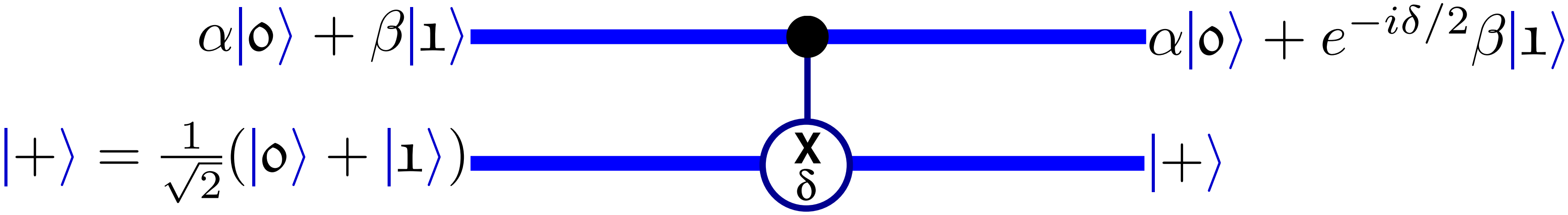

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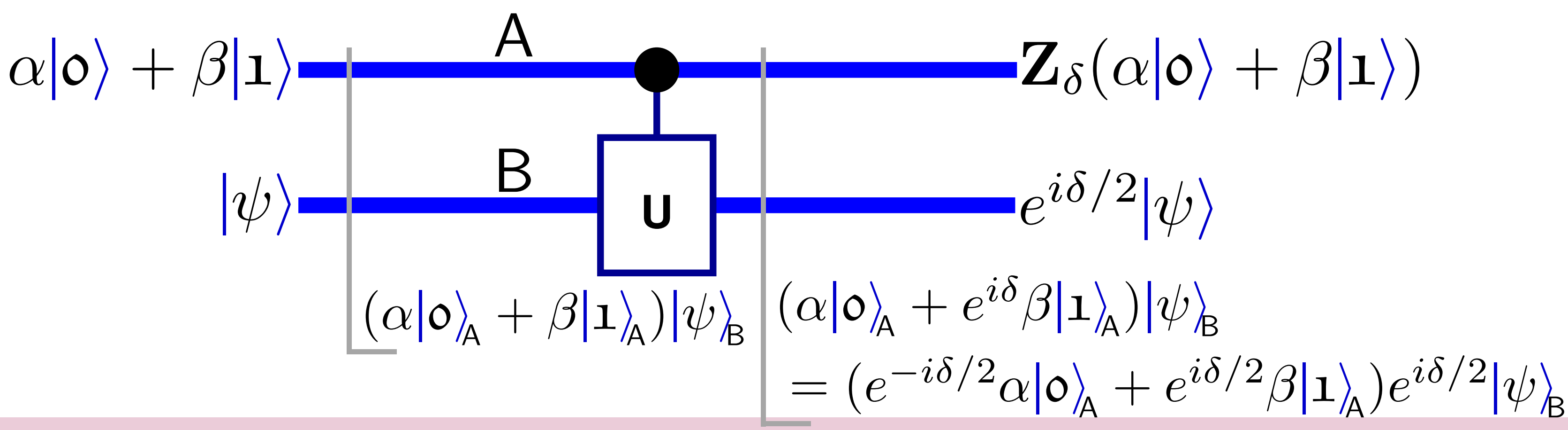
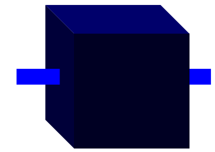


The Rotation Angle Problem (RAP)



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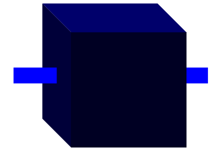

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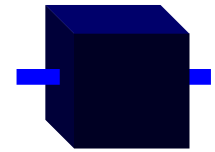

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- $O(f(\ldots))$ (“order of $f(\ldots)$”) means
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RAP by Repeat Measurements?

- Solve RAP by obtaining measurement statistics after modified queries that rotate $|0\rangle$ toward $|1\rangle$.

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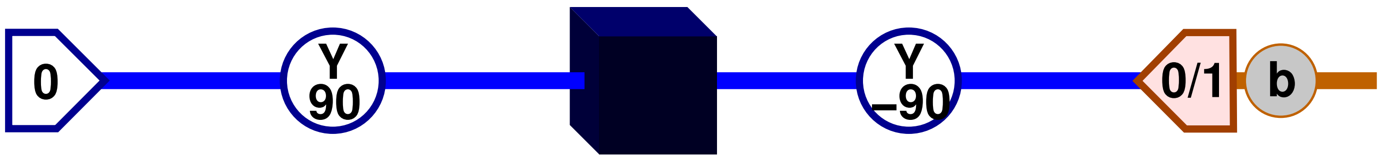

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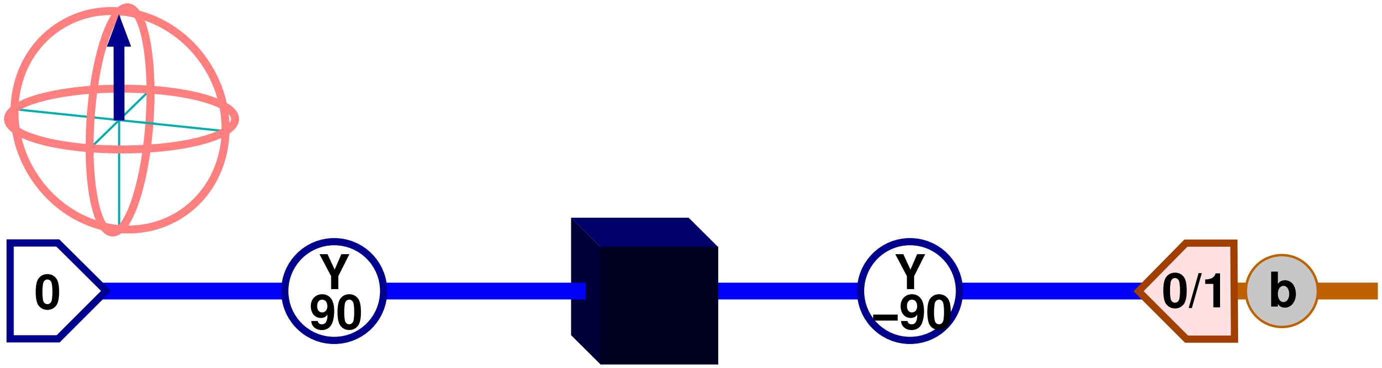

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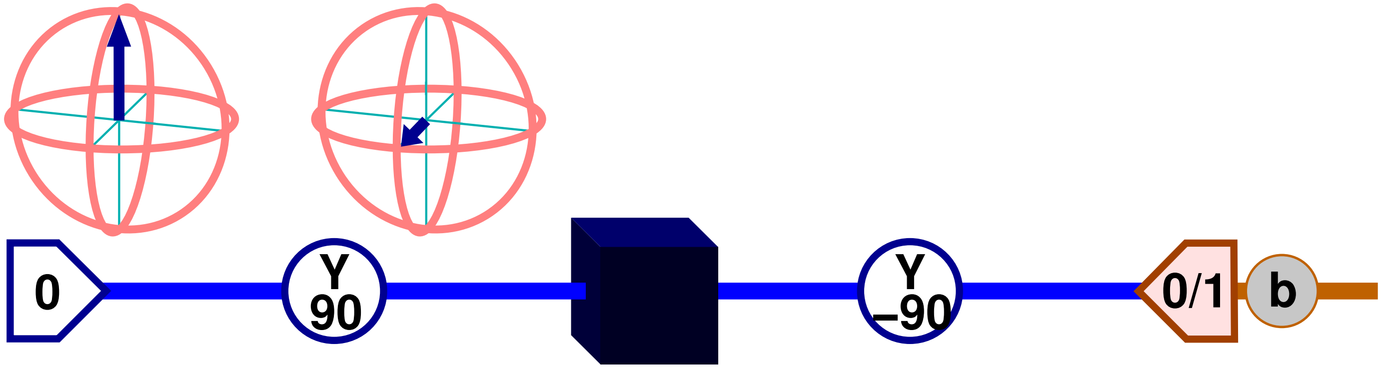

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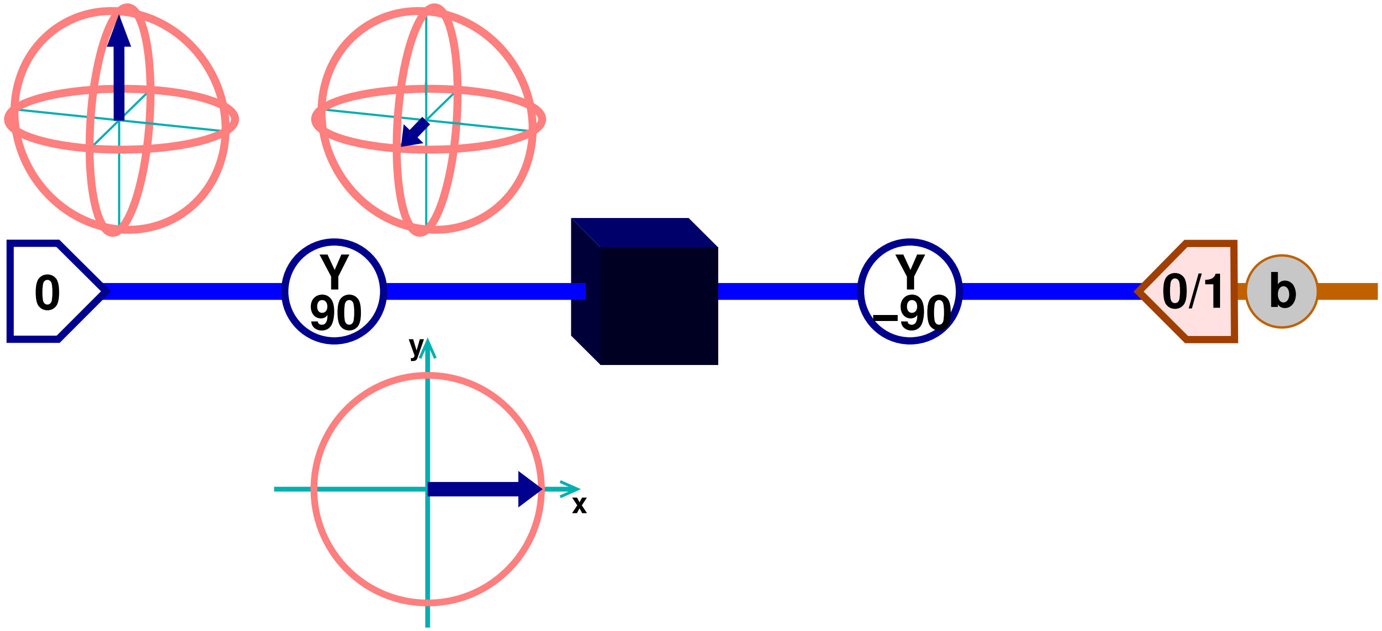

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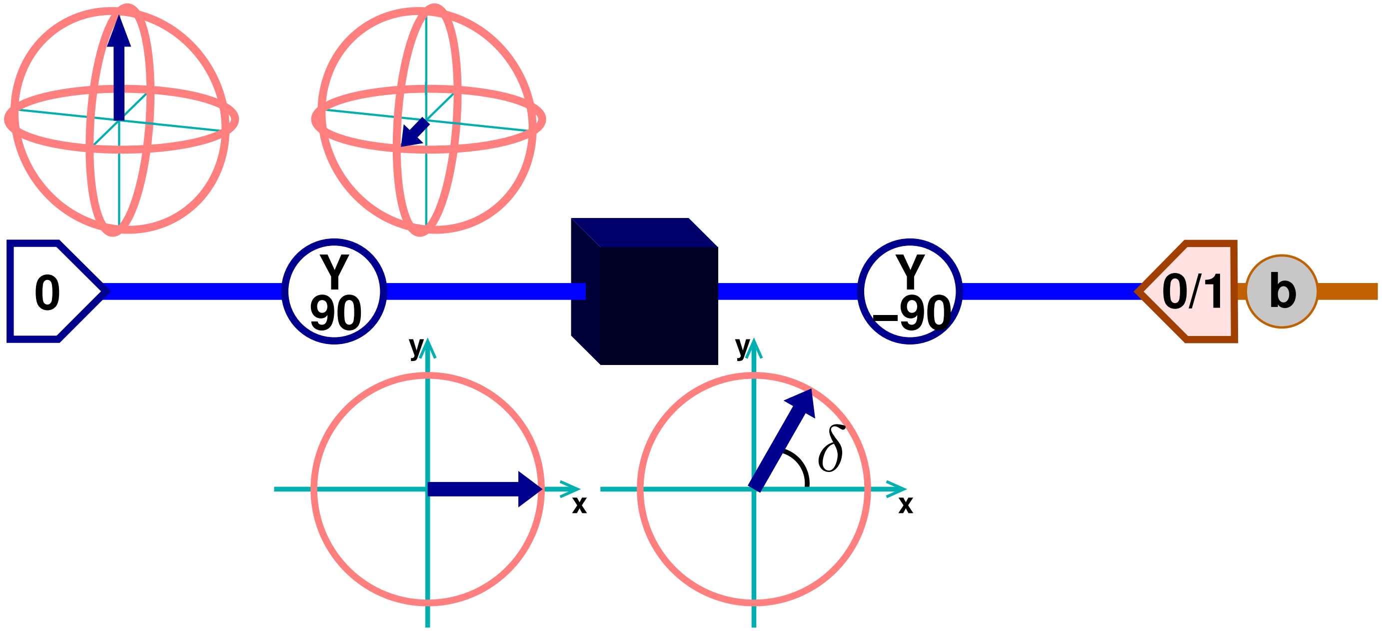

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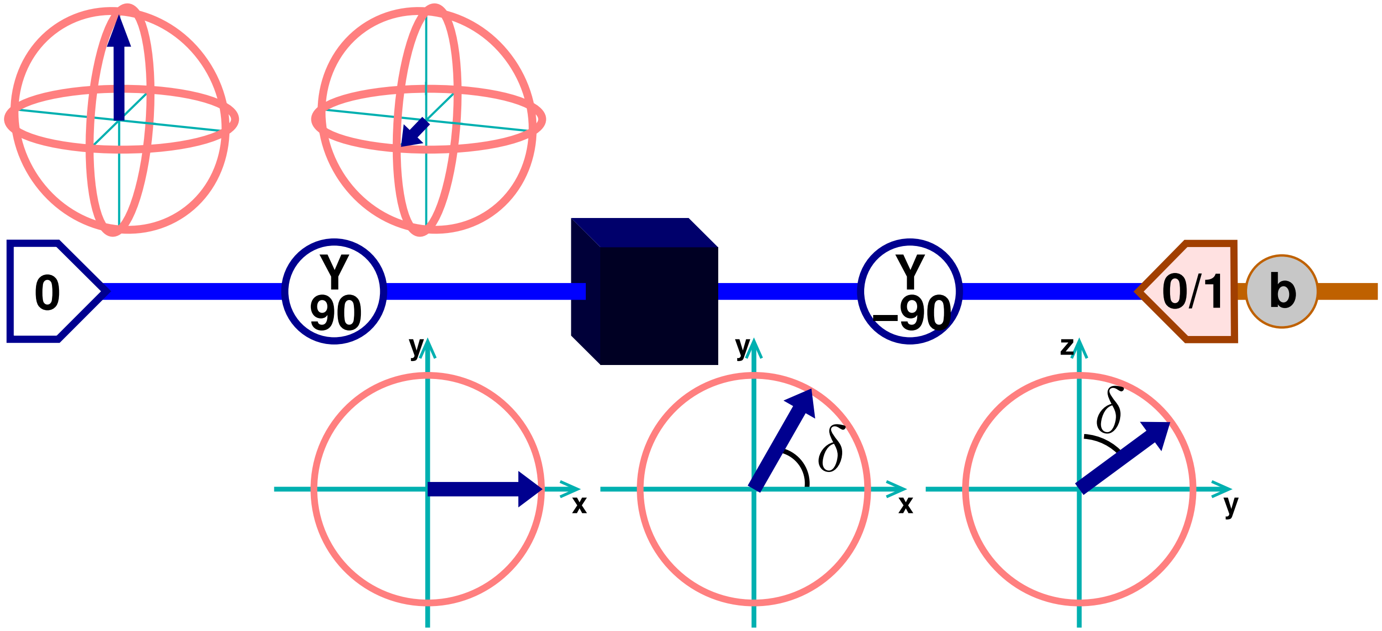

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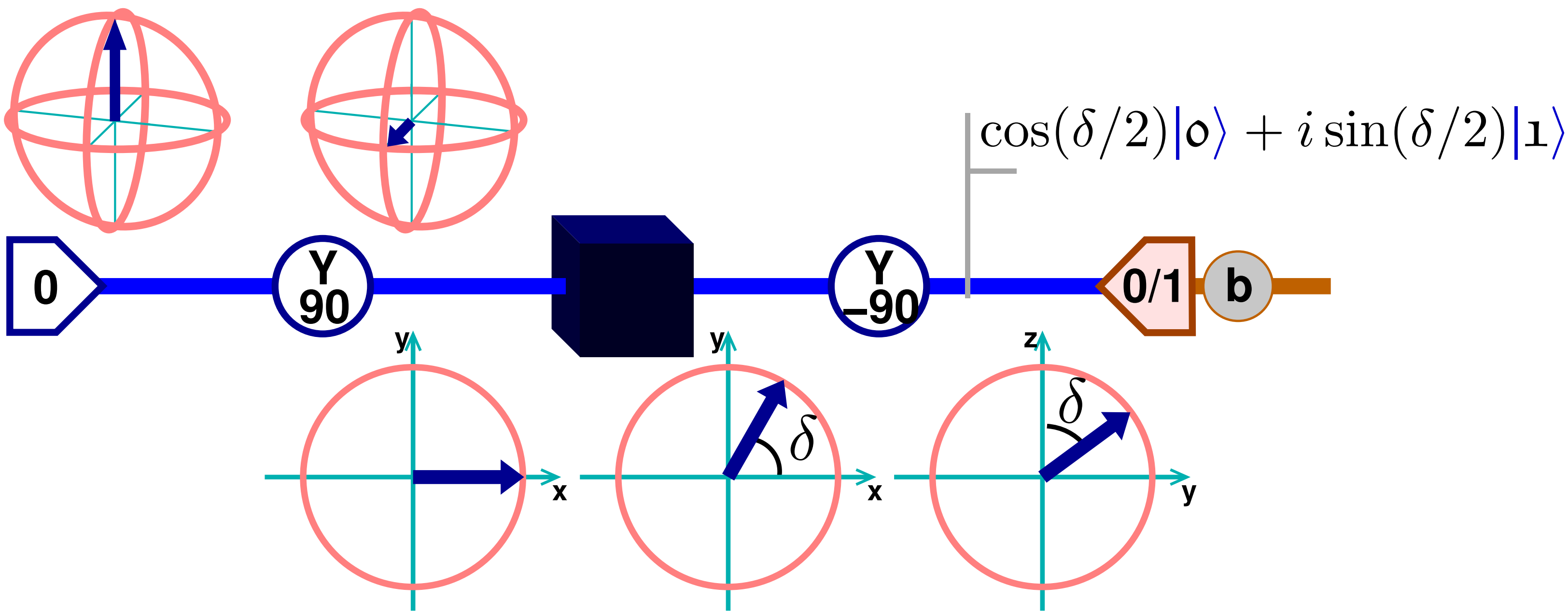

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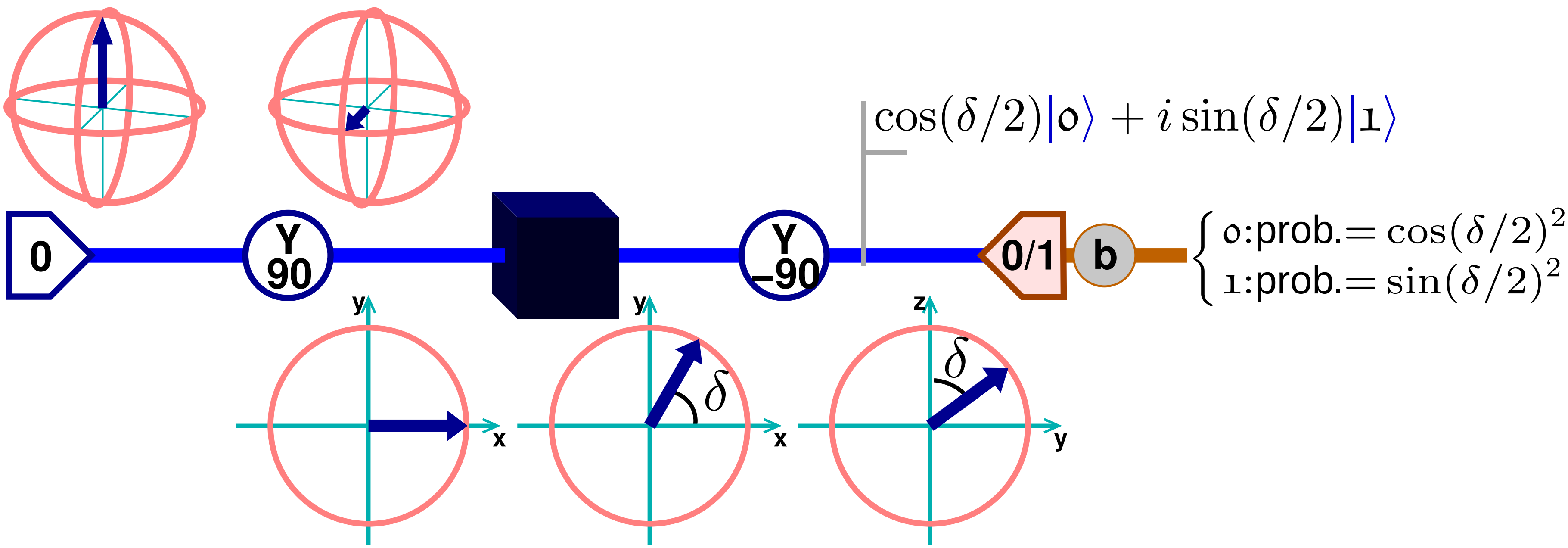

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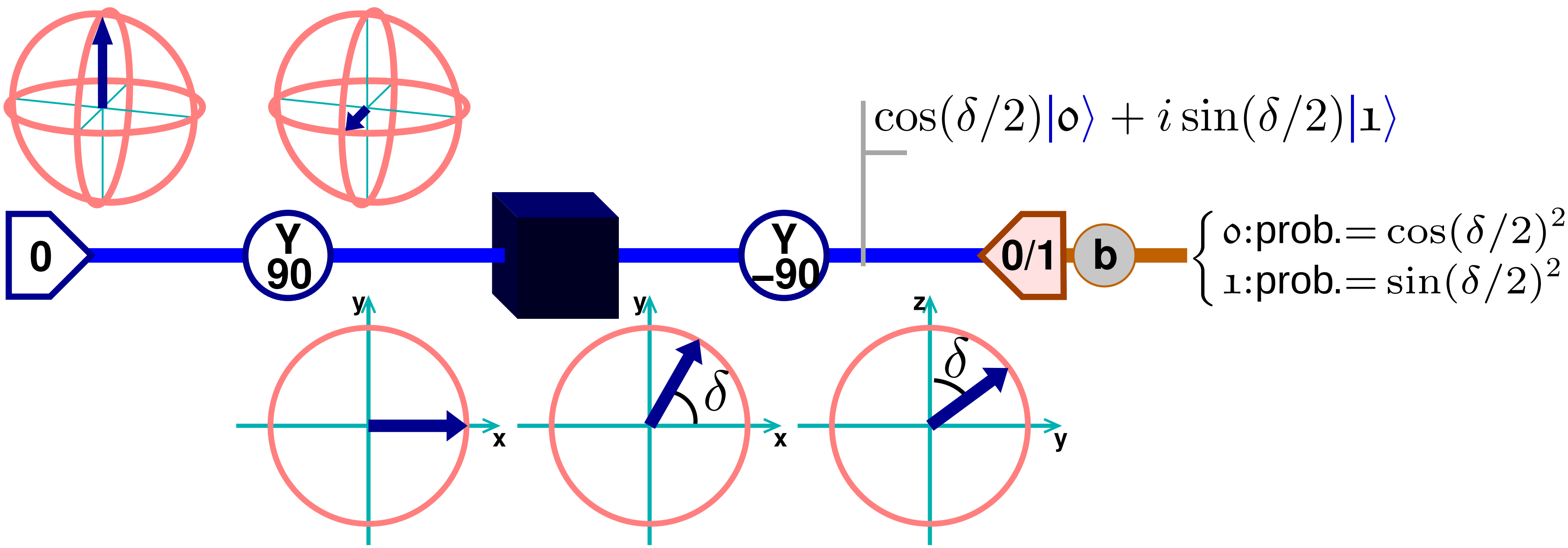

- Cannot distinguish between δ and $\delta + 180°$.

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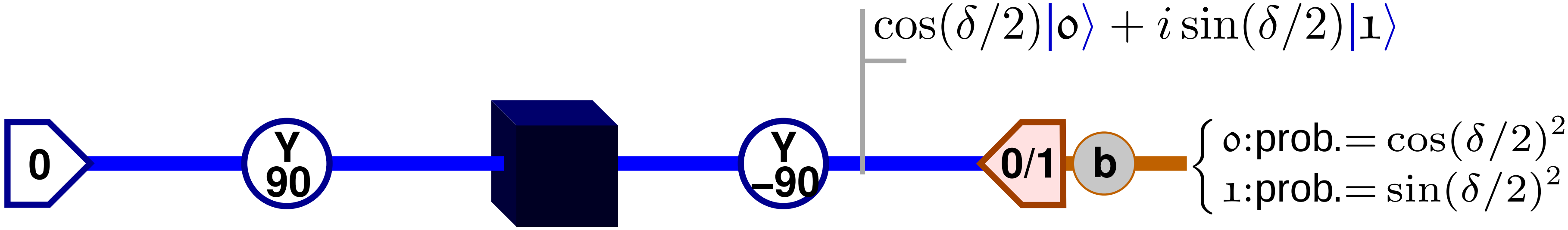

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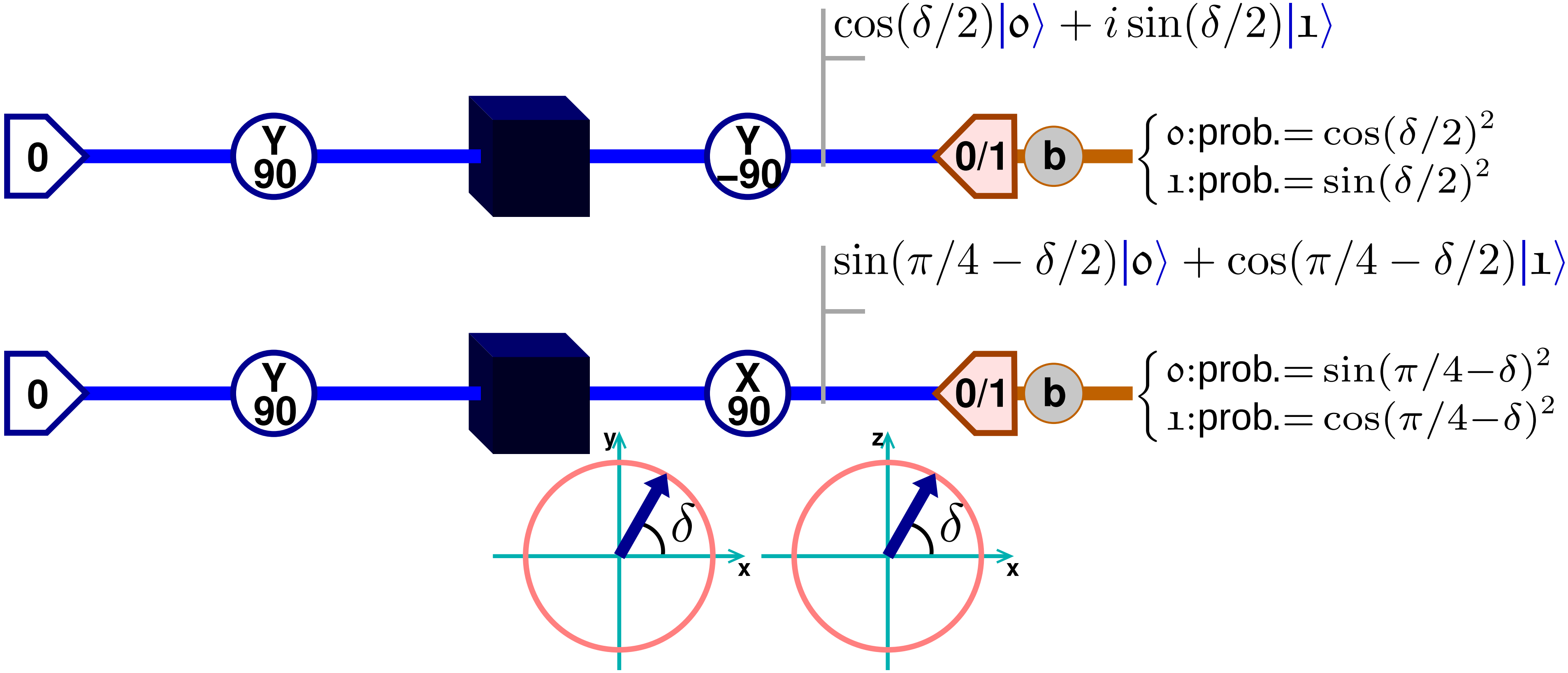

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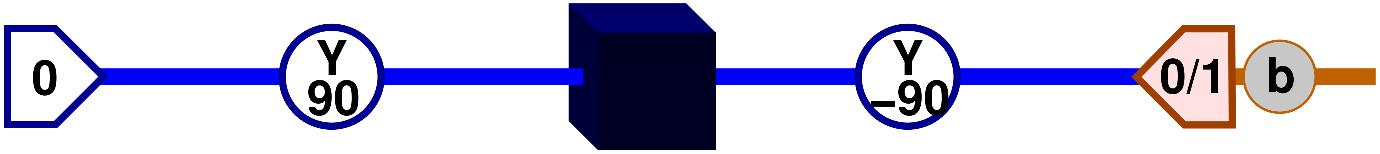
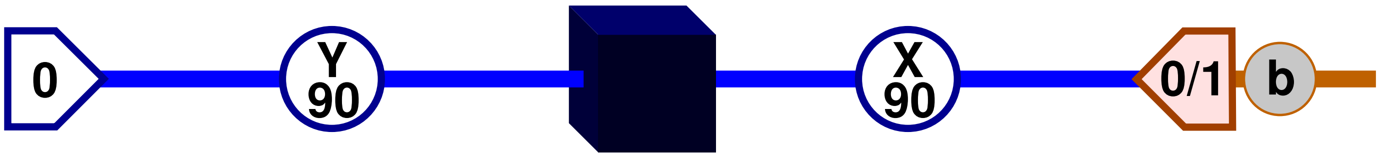


$\begin{cases} 0\text{:prob.} = \cos(\delta/2)^2 \\ 1\text{:prob.} = \sin(\delta/2)^2 \end{cases}$



$\begin{cases} 0\text{:prob.} = \sin(\pi/4-\delta/2)^2 \\ 1\text{:prob.} = \cos(\pi/4-\delta/2)^2 \end{cases}$

- Coin flip statistics for prob$(\mathfrak{b} = 1) = p$, N trials:
  Expectation: $\langle\mathfrak{b}\rangle = p$.
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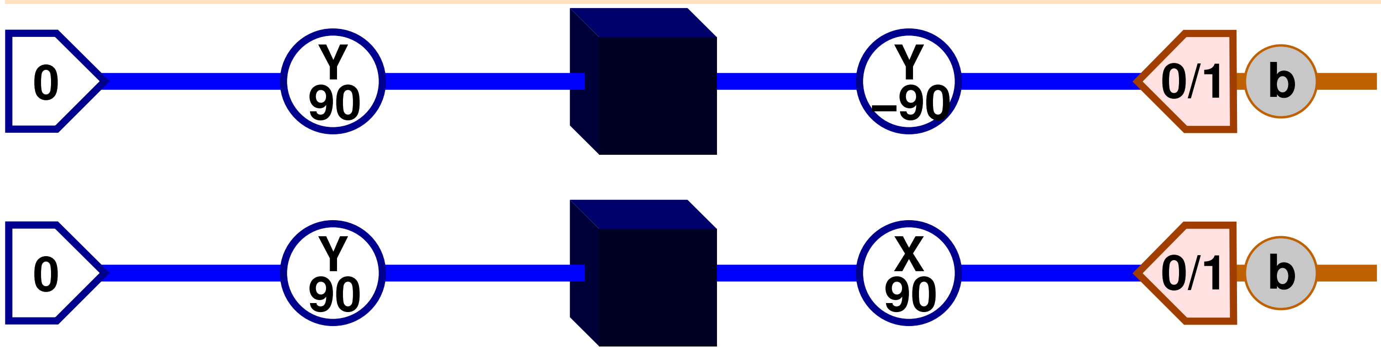

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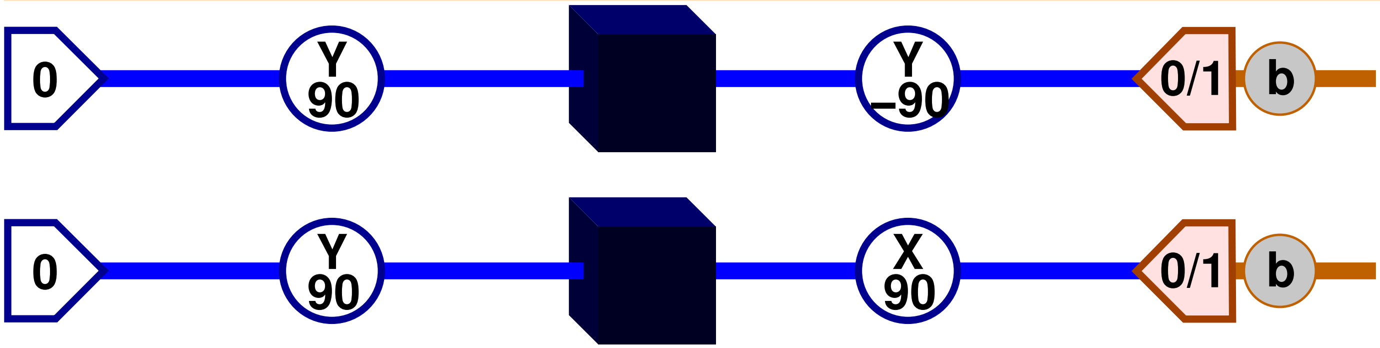
  Variance: $v = p(1-p)/N$.

- The probability that $\bar{\mathfrak{b}} = \sum_i \mathfrak{b}_i/N$ is more than Δ away from p is

$$C(\Delta) \;<\; 2e^{-\Delta^2 N/2}$$

Chernoff 1952 [1]

Measurement Statistics



$$\begin{cases} 0\text{:prob.} = \cos(\delta/2)^2 \\ 1\text{:prob.} = \sin(\delta/2)^2 \end{cases}$$

$$\begin{cases} 0\text{:prob.} = \sin(\pi/4-\delta/2)^2 \\ 1\text{:prob.} = \cos(\pi/4-\delta/2)^2 \end{cases}$$

- Coin flip statistics for prob$(\mathfrak{b} = 1) = p$, N trials:
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- From N pairs of experiments, get angle estimate $\tilde{\delta}$:
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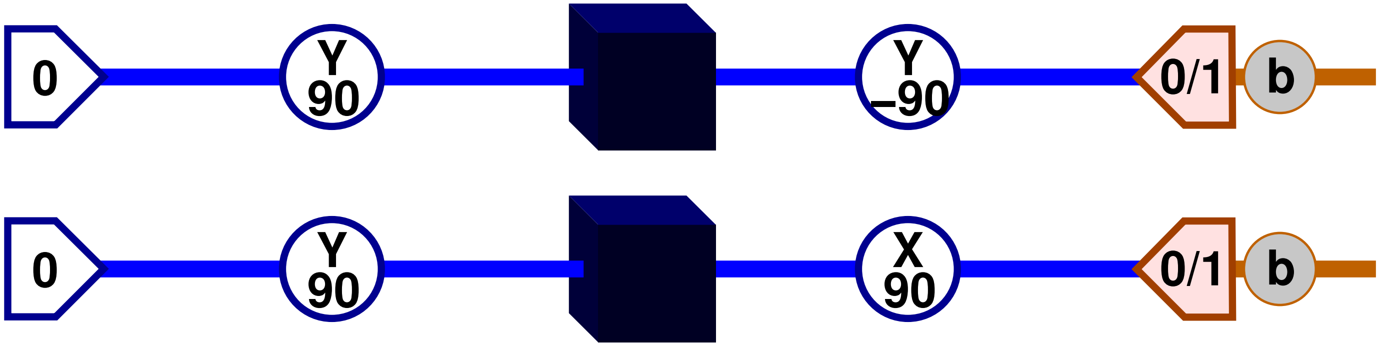
Measurement Statistics



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  $$C(\Delta) \;<\; 2e^{-\Delta^2 N/2}$$ Chernoff 1952 [1]
- From N pairs of experiments, get angle estimate $\tilde{\delta}$:
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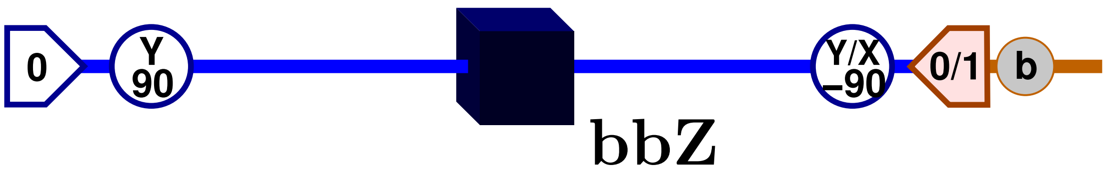
- Need to improve accuracy and reduce measurement count.

RAP by Iteration

1. Determine δ to within $\pm\pi/8$.



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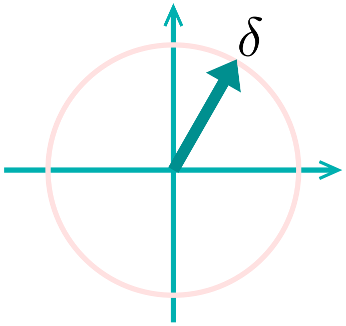

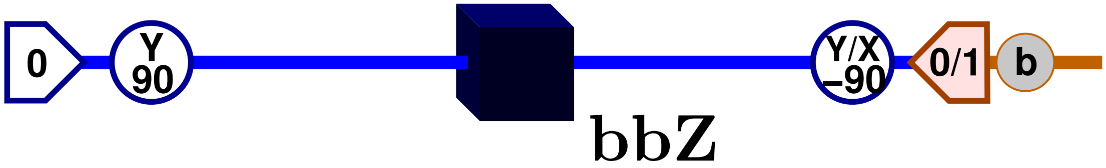

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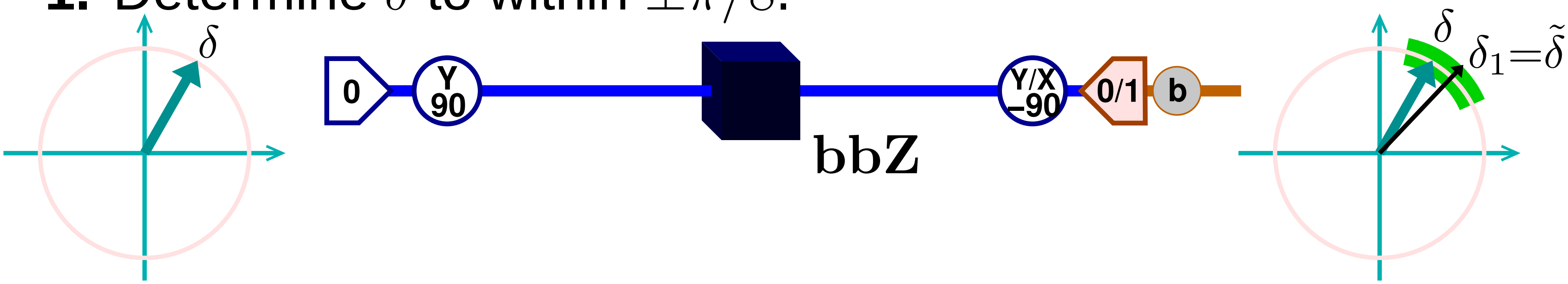

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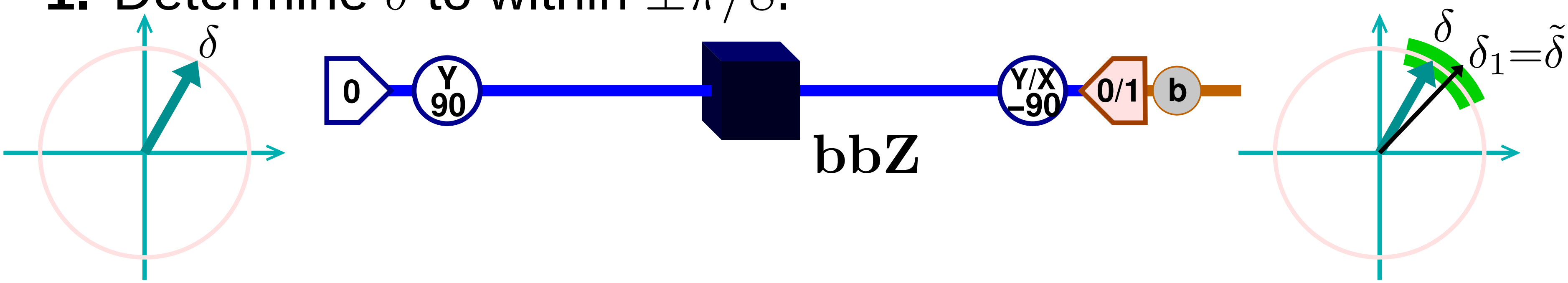

2. Use $\mathbf{bbZ}^2$ to determine δ to within $\pm\pi/16$.

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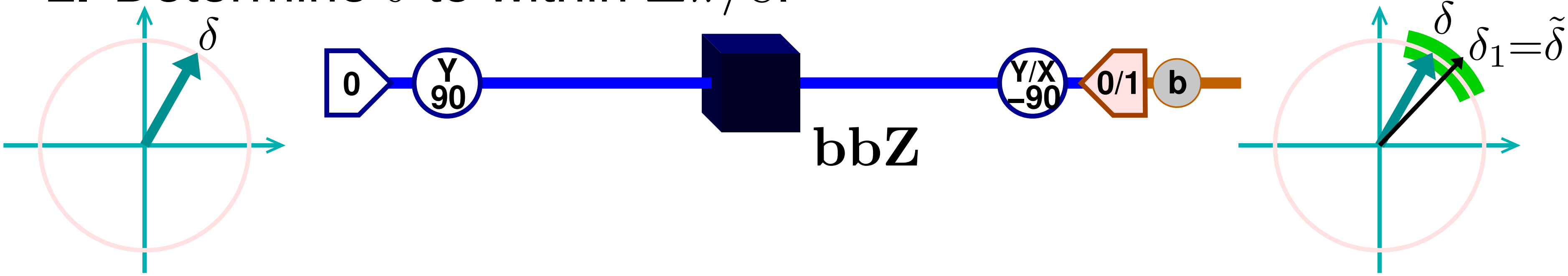

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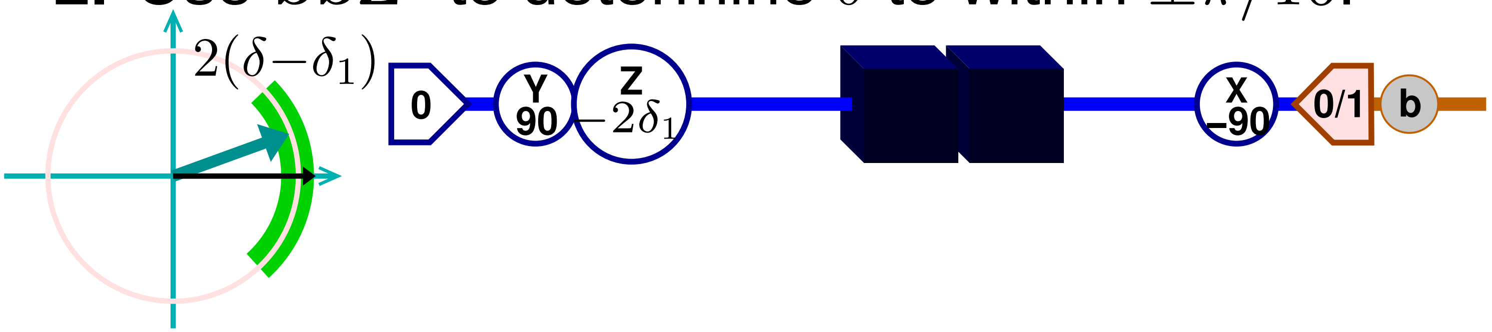

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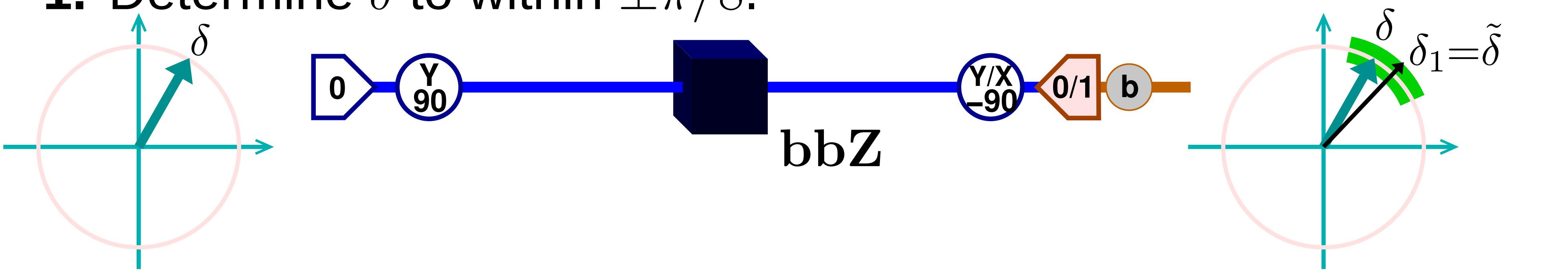

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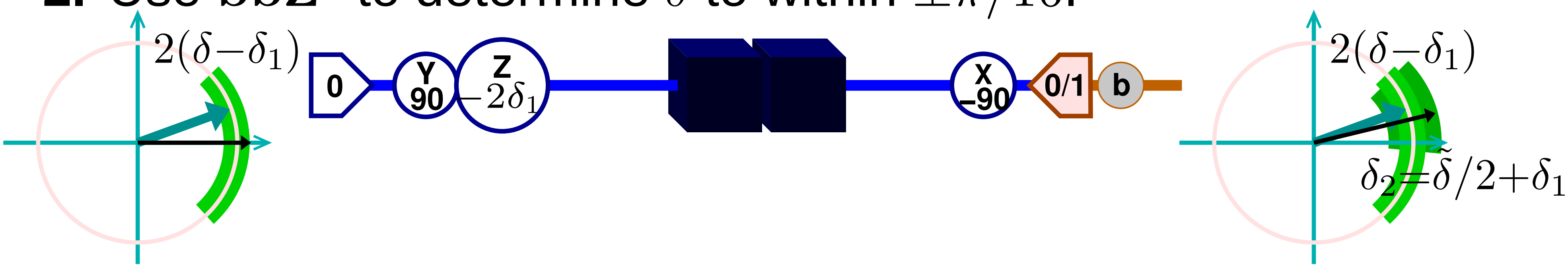

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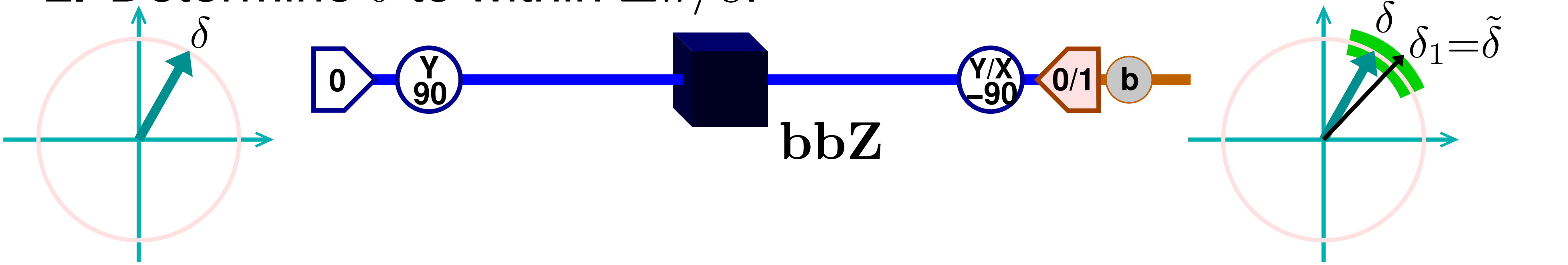

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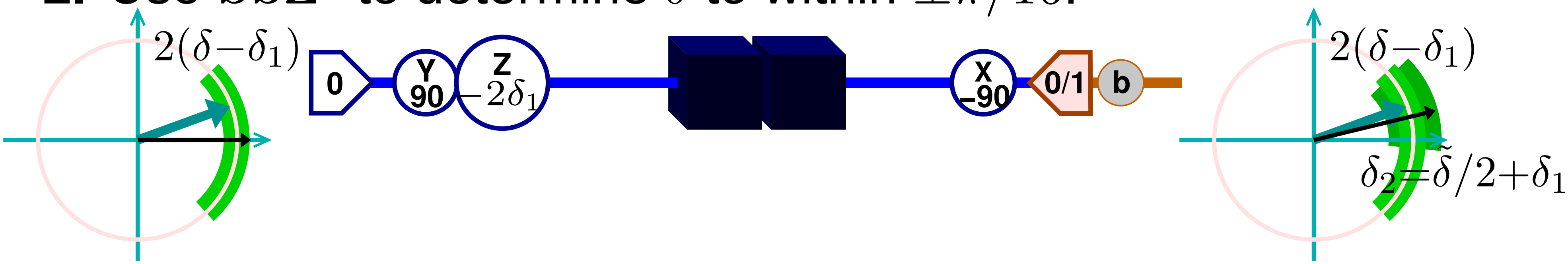
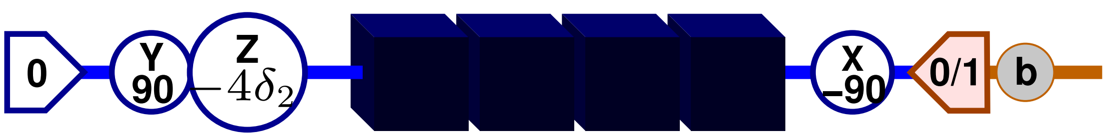


3. Use $\mathbf{bbZ}^4$ to determine δ to within $\pm\pi/32$.



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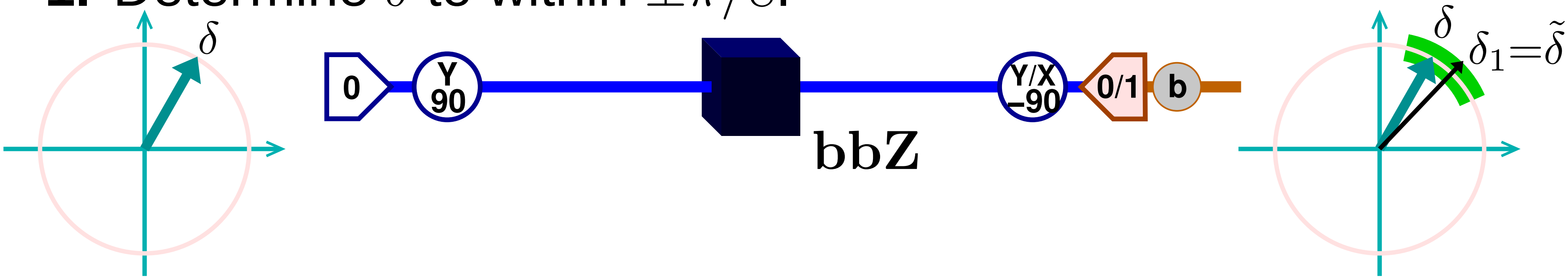

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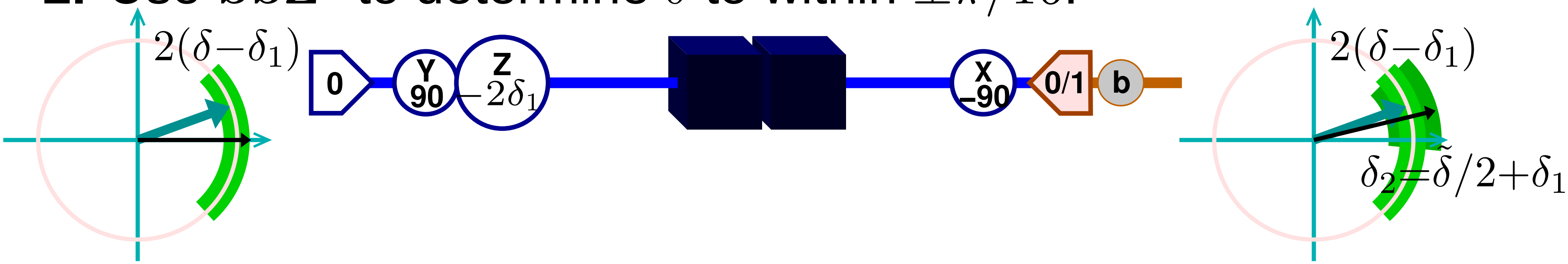

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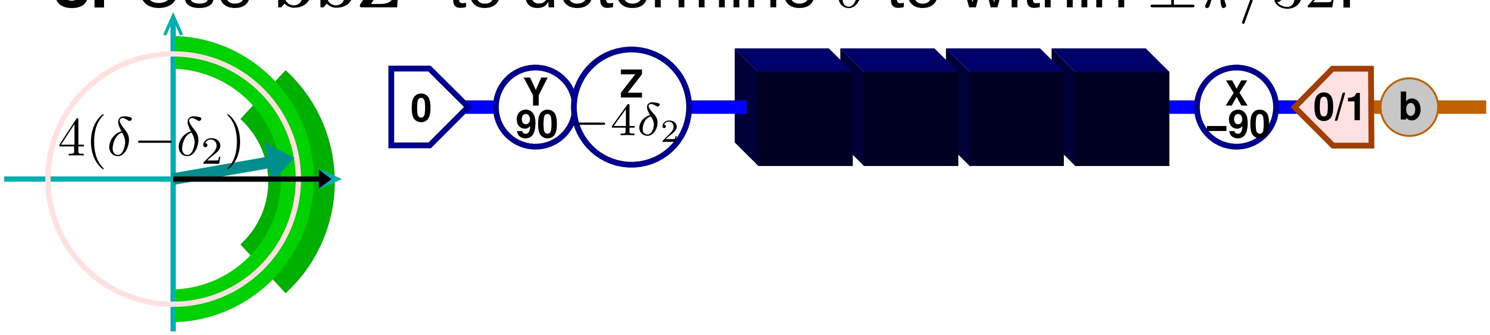

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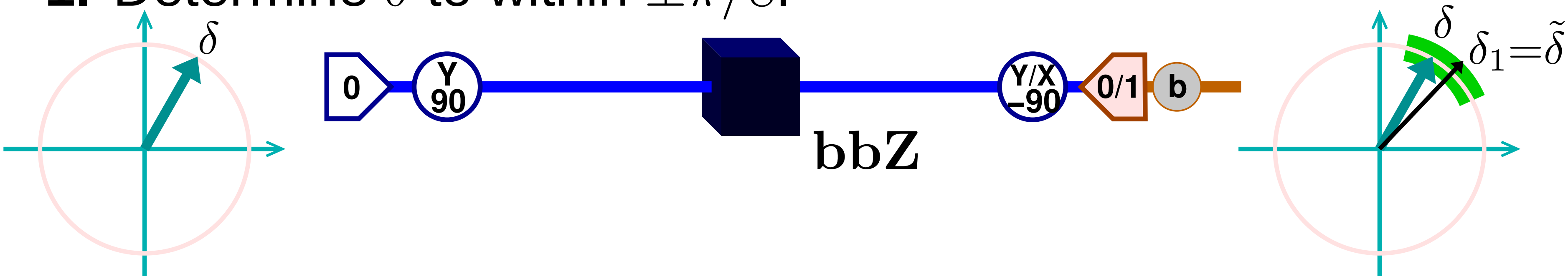

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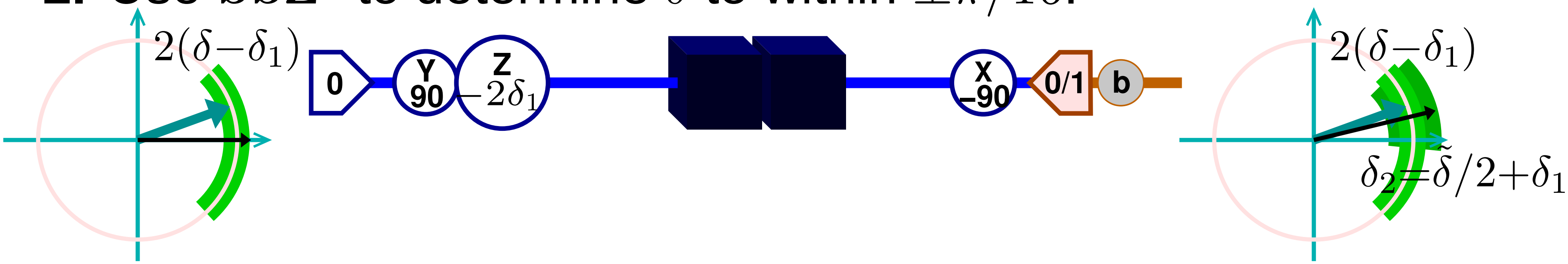

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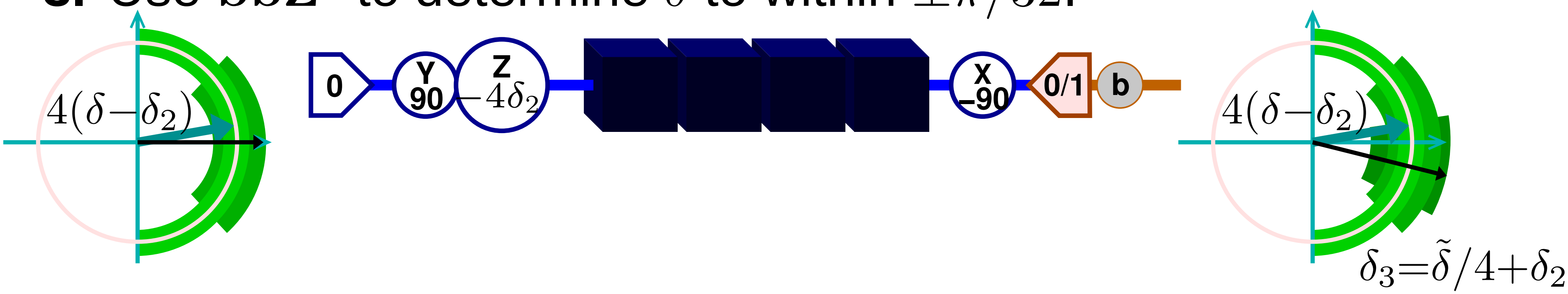

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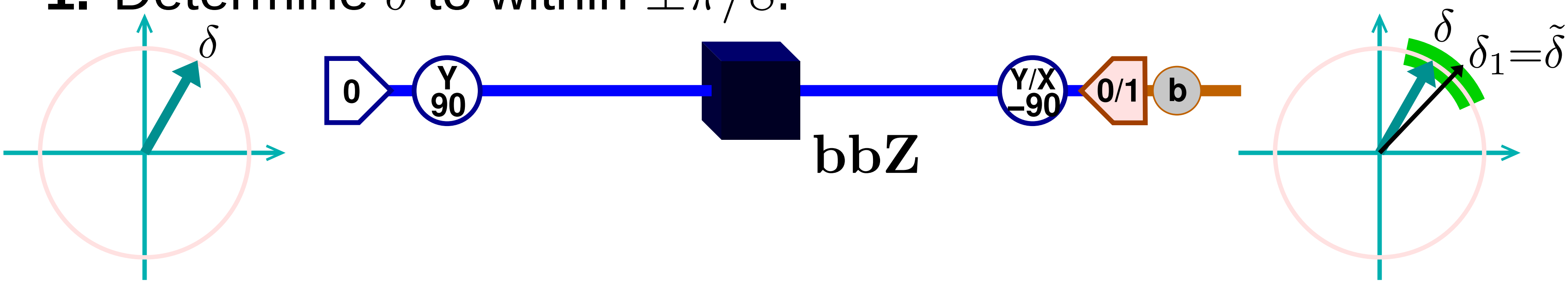

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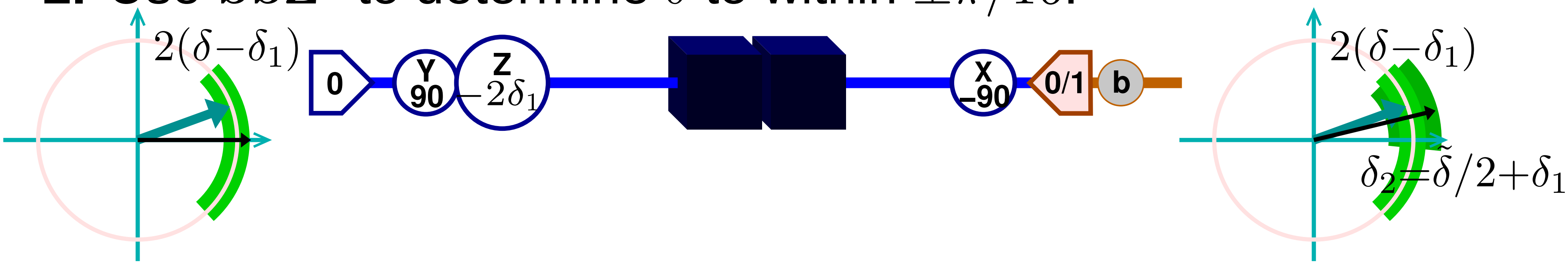

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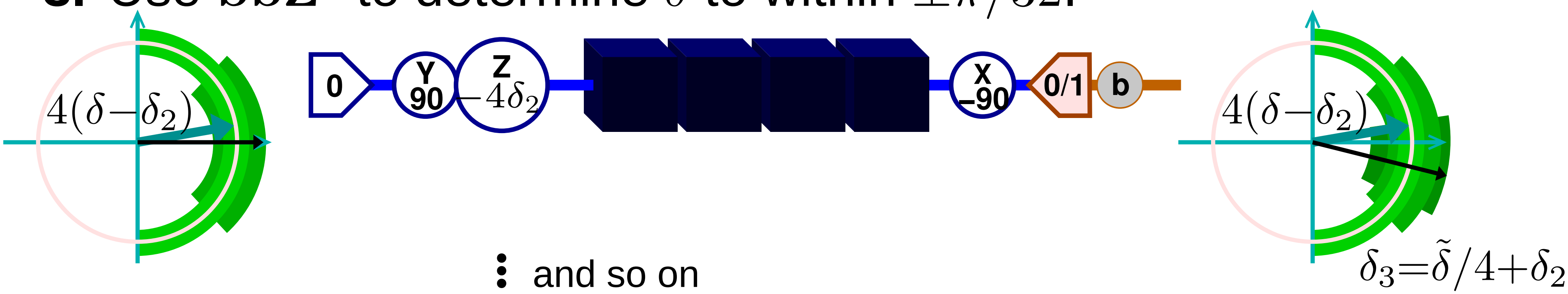
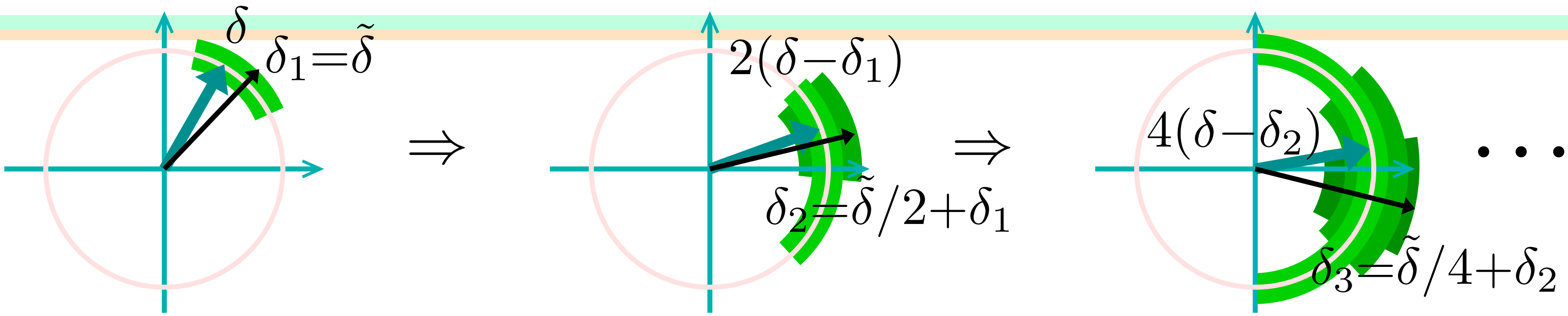


⋮ and so on

RAP by Iteration: Resources



- Let N be the number of steps.
  Let k be the number of measurements in each step.

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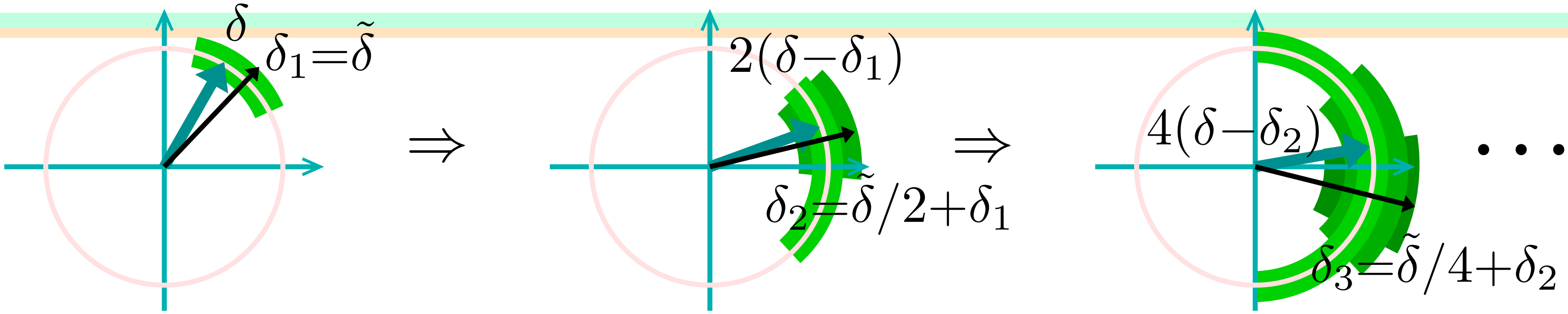

- Let N be the number of steps.
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- Approximation: Obtain δ within $\epsilon = \pi/2^{N+2}$.

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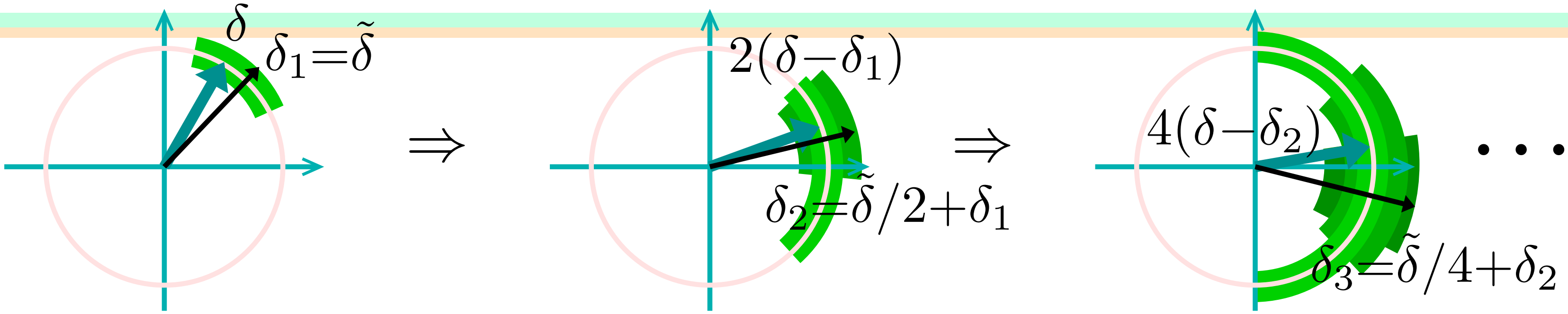

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  Confidence $1 - e^{-C}$ requires
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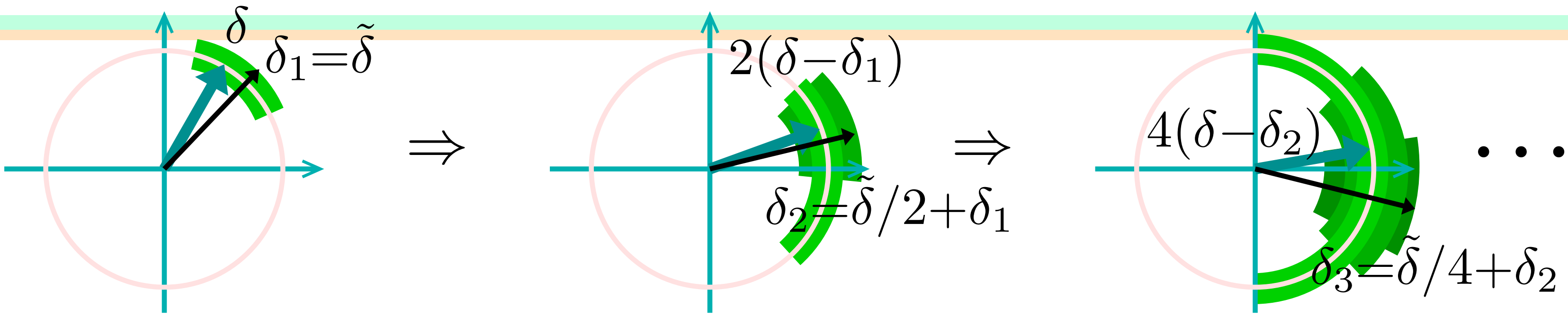
RAP by Iteration: Resources



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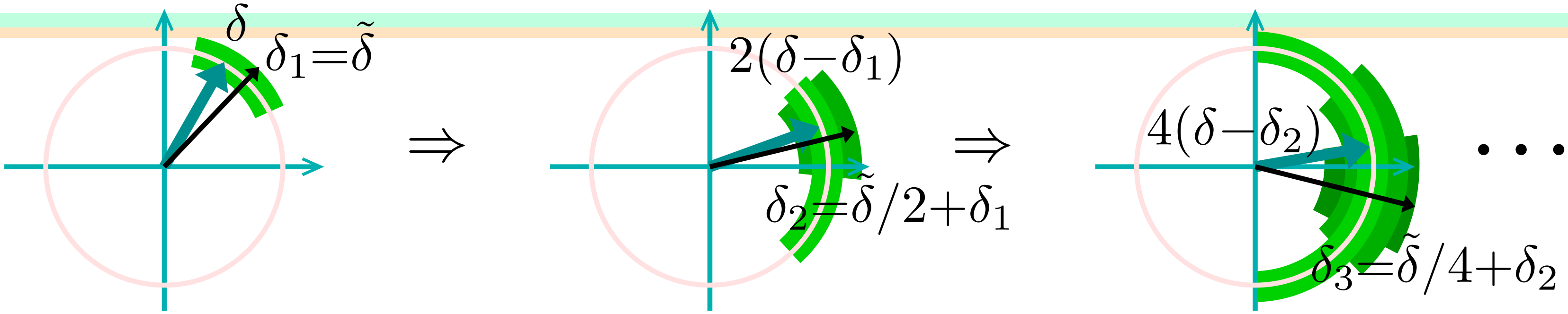

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- Number of black box queries: $< k2^{N+1} = O(\frac{1}{\epsilon}\log\log(\frac{1}{\epsilon}))$

Contents

References

[1] H. Chernoff. A measure of the asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Stat.*, 23:493–509, 1952.